French King Bridge, Greenfield, Massachusetts. The trussed arch bridge provides an efficient design to carry a roadway over a river in a rural area of Western Massachusetts. The arch configuration of the lower chord is not only visually attractive, but provides optimum headroom for boats passing under the bridge. The large depth of the structure towards the ends produces a stiff structure with slender members.
Arches

7.1 Introduction

As we discussed in Section 1.5, the arch uses material efficiently because applied loads create mostly axial compression on all cross sections. In this chapter we show that for a particular set of loads, the designer can establish one shape of arch—the funicular shape—in which all sections are in direct compression (moments are zero).

Typically, dead load constitutes the major load supported by the arch. If a funicular shape is based on the dead load distribution, moments will be created on cross sections by live loads whose distribution differs from that of the dead load. But normally in most arches, the bending stresses produced by live load moments are so small compared to the axial stresses that net compression stresses exist on all sections. Because arches use material efficiently, designers often use them as the main structural elements in long-span bridges (say, 400 to 1800 ft) or buildings that require large column-free areas, for example, airplane hangars, field houses, or convention halls.

In this chapter we consider the behavior and analysis of three-hinged arches. As part of this study, we derive the equation for the shape of a funicular arch that supports a uniformly distributed load, and we apply the general cable theory (Sec. 6.5) to produce the funicular arch for an arbitrary set of concentrated loads. Finally, we apply the concept of structural optimization to establish the minimum weight of a simple three-hinged arch carrying a concentrated load.

7.2 Types of Arches

Arches are often classified by the number of hinges they contain or by the manner in which their bases are constructed. Figure 7.1 shows the three main types: three-hinged, two-hinged, and fixed-ended. The three-hinged arch is statically determinate; the other two types are indeterminate. The three-hinged arch is the easiest to analyze and construct. Since it is determinate, temperature changes, support settlements, and fabrication errors

Figure 7.1 Types of arches: (a) three-hinged arch, stable and determinate; (b) two-hinged arch, indeterminate to the first degree; (c) fixed-end arch, indeterminate to the third degree.
do not create stresses. On the other hand, because it contains three hinges, it is more flexible than the other arch types.

Fixed-ended arches are often constructed of masonry or concrete when the base of an arch bears on rock, massive blocks of masonry, or heavy reinforced concrete foundations. Indeterminate arches can be analyzed by the flexibility method covered in Chapter 11 or more simply and rapidly by any general-purpose computer program. To determine the forces and displacements at arbitrary points along the axis of the arch using a computer, the designer treats the points as joints that are free to displace.

In long-span bridges, two main arch ribs are used to support the roadway beams. The roadway beams can be supported either by tension hangers from the arch (Fig. 19a) or by columns that bear on the arch (Photo 7.1). Since the arch rib is mostly in compression, the designer must also consider the possibility of its buckling—particularly if it is slender (Fig. 7.2a). If the arch is constructed of steel members, a built-up rib or a box

Figure 7.2: (a) Buckling of an unsupported arch; (b) trussed arch, the vertical and diagonal members brace the arch rib against buckling in the vertical plane; (c) two types of built-up steel cross sections used to construct an arch rib.
section may be used to increase the bending stiffness of the cross section and to reduce the likelihood of buckling. In many arches, the floor system or wind bracing is used to stiffen the arch against lateral buckling. In the case of the trussed arch shown in Figure 7.2b, the vertical and diagonal members brace the arch rib against buckling in the vertical plane.

Since many people find the arch form aesthetically pleasing, designers often use low arches to span small rivers or roads in parks and other public places. At sites where rock sidewalls exist, designers often construct short-span highway bridges using barrel arches (see Fig. 7.3). Constructed of accurately fitted masonry blocks or reinforced concrete, the barrel arch consists of a wide, shallow arch that supports a heavy, compacted fill on which the engineer places the roadway slab. The large weight of the fill induces sufficient compression in the barrel arch to neutralize any tensile bending stresses created by even the heaviest vehicles. Although the loads supported by the barrel arch may be large, direct stresses in the arch itself are typically low—on the order of 300 to 500 psi because the cross-sectional area of the arch is large. A study by the senior author of a number of masonry barrel-arch bridges built in Philadelphia in the mid-nineteenth century showed that they have the capacity to support vehicles three to five times heavier than the standard AASHO truck (see Fig. 2.7), which highway bridges are currently designed to support. Moreover, while many steel and reinforced concrete bridges built in the past 100 years are no longer serviceable because of corrosion produced by salts used to melt snow, many masonry arches, constructed of good-quality stone, show no deterioration.

### 7.3 Three-Hinged Arches

To demonstrate certain of the characteristics of arches, we will consider how the bar forces vary as the slope of the bars changes in the pin-jointed arch in Figure 7.4a. Since the members carry axial load only, this configuration represents the funicular shape for an arch supporting a single concentrated load at midspan.

![Photo 7.1: Railroad bridge (1909) over the Landwasser Gorge, near Wiesen, Switzerland. Masonry construction. The main arch is parabolic, has a span of 55 m and a rise of 33 m. The bridge is narrow as the railway is single-track. The arch ribs are a mere 4.8 m at the crown, tapering to 6 m at the supports.](image)

![Figure 7.3: (a) Barrel arch resembles a curved slab; (b) barrel arch used to support a compacted fill and roadway slab.](image)

![Figure 7.4: (a) Three-hinged arch with a concentrated load; (b) vector diagram of forces acting on the hinge at B, forces $F_{CB}$ and $F_{AB}$ are equal because of symmetry; (c) components of force in bar AB.](image)
Because of symmetry, the vertical components of the reactions at supports \( A \) and \( C \) are identical in magnitude and equal to \( P/2 \). Denoting the slope of bars \( AB \) and \( CB \) by angle \( \theta \), we can express the bar forces \( F_{AB} \) and \( F_{CB} \) in terms of \( P \) and the slope angle \( \theta \) (see Fig. 7.4b) as

\[
\sin \theta = \frac{P}{2} = \frac{F_{AB}}{F_{CB}}
\]

\[
F_{AB} = F_{CB} = \frac{P}{2 \sin \theta} \tag{7.1}
\]

Equation 7.1 shows that as \( \theta \) increases from 0 to 90°, the force in each bar decreases from infinity to \( P/2 \). We can also observe that as the slope angle \( \theta \) increases, the length of the bars—and consequently the material required—also increases. To establish the slope that produces the most economical structure for a given span \( L \), we will express the volume \( V \) of bar material required to support the load \( P \) in terms of the geometry of the structure and the compressive strength of the material

\[
V = 2AL_B \tag{7.2}
\]

where \( A \) is the area of one bar and \( L_B \) is the length of a bar.

To express the required area of the bars in terms of load \( P \), we divide the bar forces given by Equation 7.1 by the allowable compressive stress \( f_{allow} \):

\[
A = \frac{P}{2 \sin \theta \cdot f_{allow}} \tag{7.3}
\]

We will also express the bar length \( L_B \) in terms of \( \theta \) and the span length \( L \) as

\[
L_B = \frac{L}{2 \cos \theta} \tag{7.4}
\]

Substituting \( A \) and \( L_B \) given by Equations 7.3 and 7.4 into Equation 7.2, simplifying, and using the trigonometric identity \( \sin 2\theta = 2 \sin \theta \cdot \cos \theta \), we calculate

\[
V = \frac{PL}{2 \cdot f_{allow} \sin 2\theta} \tag{7.5}
\]

If \( V \) in Equation 7.5 is plotted as a function of \( \theta \) (see Fig. 7.5), we observe that the minimum volume of material is associated with an angle of \( \theta = 45^\circ \). Figure 7.5 also shows that very shallow arches (\( \theta \leq 15^\circ \)) and very deep arches (\( \theta \geq 75^\circ \)) require a large volume of material; on the other hand, the flat curvature in Figure 7.5 when \( \theta \) varies between 30 and 60° indicates that the volume of the bars is not sensitive to the slope between these limits. Therefore, the designer can vary the shape of the structure within this range without significantly affecting either its weight or its cost.
In the case of a curved arch carrying a distributed load, the engineer will also find that the volume of material required in the structure, within a certain range, is not sensitive to the depth of the arch. Of course, the cost of a very shallow or very deep arch will be greater than that of an arch of moderate depth. Finally, in establishing the shape of an arch, the designer will also consider the profile of the site, the location of solid bearing material for the foundations, and the architectural and functional requirements of the project.

### 7.4 Funicular Shape for an Arch That Supports a Uniformly Distributed Load

Many arches carry dead loads that have a uniform or nearly uniform distribution over the span of the structure. For example, the weight per unit length of the floor system of a bridge will typically be constant. To establish for a uniformly loaded arch the funicular shape—the form required if only direct stress is to develop at all points along the axis of an arch—we will consider the symmetric three-hinged arch in Figure 7.6a. The height (or rise) of the arch is denoted by \( h \). Because of symmetry, the vertical reactions at supports \( A \) and \( C \) are equal to \( wL/2 \) (one-half the total load supported by the structure).

The horizontal thrust \( H \) at the base of the arch can be expressed in terms of the applied load \( w \) and the geometry of the arch by considering the free body to the right of the center hinge in Figure 7.6b. Summing moments about the center hinge at \( B \), we find

\[
\frac{1}{2} \left( \frac{L}{2} - x \right) \left( \frac{w}{2} \right) = R \left( \frac{L}{2} - x \right) - \frac{wL}{2}
\]

where \( R \) is the reaction at \( B \). Solving for \( R \), we get

\[
R = w \left( \frac{L}{2} - x \right)
\]

Figure 7.6: Establishing the funicular shape for a uniformly loaded arch.
\[ \sum M_B = 0 \]
\[ 0 = \frac{wL}{2} \frac{L}{4} - \frac{wL}{2} \frac{L}{2} + Hh \]
\[ H = \frac{wL^2}{8h} \]

(7.6)

To establish the equation of the axis of the arch, we superimpose a rectangular coordinate system, with an origin \( o \) located at \( B \), on the arch. The positive sense of the vertical \( y \) axis is directed downward. We next express the moment \( M \) at an arbitrary section (point \( D \) on the arch’s axis) by considering the free body of the arch between \( D \) and the pin at \( C \).

\[ \sum M_D = 0 \]
\[ 0 = \frac{L}{2} - x \cdot \frac{w}{2} - \frac{wL}{2} \frac{L}{2} - x \cdot H \cdot h - y \cdot M \]

Solving for \( M \) gives
\[ M = \frac{wL^2y}{8h} - \frac{wx^2}{2} \]

(7.7)

If the arch axis follows the funicular shape, \( M = 0 \) at all sections. Substituting this value for \( M \) into Equation 7.7 and solving for \( y \) establishes the following mathematical relationship between \( y \) and \( x \):
\[ y = \frac{4h}{L^2} x^2 \]

(7.8)

Equation 7.8, of course, represents the equation of a parabola. Even if the parabolic arch in Figure 7.6 were a fixed-ended arch, a uniformly distributed load—assuming no significant change in geometry from axial shortening—would still produce direct stress at all sections because the arch conforms to the funicular shape for a uniform load.

From a consideration of equilibrium in the horizontal direction, we can see that the horizontal thrust at any section of an arch equals \( H \), the horizontal reaction at the support. In the case of a uniformly loaded parabolic arch, the total axial thrust \( T \) at any section, a distance \( x \) from the origin at \( B \) (see Fig. 7.6b), can be expressed in terms of \( H \) and the slope at the given section as
\[ T = \frac{H}{\cos \cdot} \]

(7.9)

To evaluate \( \cos \cdot \), we first differentiate Equation 7.8 with respect to \( x \) to give
\[ \tan \cdot = \frac{dy}{dx} = \frac{8hx}{L^2} \]

(7.10)
The tangent of \( \phi \) can be shown graphically by the triangle in Figure 7.6c.
From this triangle we can compute the hypotenuse \( r \) using \( r^2 = x^2 + y^2 \):

\[
r = \sqrt{1 + \frac{8h_2}{L^2}}
\]  
(7.11)

From the relationship between the sides of the triangle in Figure 7.6c and the cosine function, we can write

\[
\cos \phi = \frac{1}{\sqrt{1 + \frac{8h_2}{L^2}}}
\]  
(7.12)

Substituting Equation 7.12 into Equation 7.9 gives

\[
T = H \cdot \frac{1}{\sqrt{1 + \frac{8h_2}{L^2}}}
\]  
(7.13)

Equation 7.13 shows that the largest value of thrust occurs at the supports where \( \phi \) has its maximum value of \( L/2 \). If \( w \) or the span of the arch is large, the designer may wish to vary (taper) the cross section in direct proportion to the value of \( T \) so that the stress on the cross section is constant.

Example 7.1 illustrates the analysis of a three-hinged trussed arch for both a set of loads that corresponds to the funicular shape of the arch as well as for a single concentrated load. Example 7.2 illustrates the use of cable theory to establish a funicular shape for the set of vertical loads in Example 7.1.

---

**EXAMPLE 7.1**

Analyze the three-hinged trussed arch in Figure 7.7a for the dead loads applied at the top chord. Member \( KJ \), which is detailed so that it cannot transmit axial force, acts as a simple beam instead of a member of the truss. Assume joint \( D \) acts as a hinge.

**Solution**
Because the arch and its loads are symmetric, the vertical reactions at \( A \) and \( G \) are equal to 180 kips (one-half the applied load). Compute the horizontal reaction at support \( G \).

Consider the free body of the arch to the right of the hinge at \( D \) (Fig. 7.7a), and sum moments about \( D \).

\[
\sum M_D = 0 \Rightarrow 0 = 60 \cdot 30 \cdot + 60 \cdot 60 + 30 \cdot 90 \cdot - 180 \cdot 90 \cdot + 36H
\]
\[H = 225 \text{ kips}
\]

[continues on next page]
Example 7.1 continues . . .

Figure 7.7

We now analyze the truss by the method of joints starting at support A. Results of the analysis are shown on a sketch of the truss in Figure 7.7b.

NOTE. Since the arch rib is the funicular shape for the loads applied at the top chord, the only members that carry load—other than the rib—are the vertical columns, which transmit the load down to the arch. The diagonals and top chords will be stressed when a loading pattern that does not conform to the funicular shape acts. Figure 7.8 shows the forces produced in the same truss by a single concentrated load at joint L.

Figure 7.8

Example 7.2

Establish the shape of the funicular arch for the set of loads acting on the trussed arch in Figure 7.7. The rise of the arch at midspan is set at 36 ft.

Solution

We imagine that the set of loads is applied to a cable that spans the same distance as the arch (see Fig. 7.9a). The sag of the cable is set at 36 ft—
the height of the arch at midspan. Since the 30-kip loads at each end of the span act directly at the supports, they do not affect the force or the shape of the cable and may be neglected. Applying the general cable theory, we imagine that the loads supported by the cable are applied to an imaginary simply supported beam with a span equal to that of the cable (Fig. 7.9b). We next construct the shear and moment curves. According to the general cable theorem at every point,

$$M = H \gamma$$

(6.6)

![Diagram of the arch with 30 kips loads at each end, cable forces, and moment curves.](diagram)

**Figure 7.9:** Use of cable theory to establish the funicular shape of an arch.

[continues on next page]
Example 7.2 continues... where \( M \) = moment at an arbitrary point in the beam  
\( H \) = horizontal component of support reaction  
\( y \) = cable sag at an arbitrary point

Since \( y = 36 \) ft at midspan and \( M = 8100 \) kip-ft, we can apply Equation 6.6 at that point to establish \( H \).

\[
H = \frac{M}{y} = \frac{8100}{36} = 225 \text{ kips}
\]

With \( H \) established we next apply Equation 6.6 at 30 and 60 ft from the supports. Compute \( y_1 \) at 30 ft:

\[
y_1 = \frac{M}{H} = \frac{4500}{225} = 20 \text{ ft}
\]

Compute \( y_2 \) at 60 ft:

\[
y_2 = \frac{M}{H} = \frac{7200}{225} = 32 \text{ ft}
\]

A cable profile is always a funicular structure because a cable can only carry direct stress. If the cable profile is turned upside down, a funicular arch is produced. When the vertical loads acting on the cable are applied to the arch, they produce compression forces at all sections equal in magnitude to the tension forces in the cable at the corresponding sections.

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**Summary**

- Although short masonry arches are often used in scenic locations because of their attractive form, they also produce economical designs for long-span structures that (1) support large, uniformly distributed dead load and (2) provide a large unobstructed space under the arch (suitable for convention halls or sports arenas or in the case of a bridge providing clearance for tall ships).
- Arches can be shaped (termed a funicular arch) so that dead load produces only direct stress—a condition that leads to a minimum weight structure.
- For a given set of loads, the funicular shape of arch can be established using cable theory.
**PROBLEMS**

**P7.1.** For the parabolic arch in Figure P7.1, plot the variation of the thrust $T$ at support $A$ for values of $h = 12$, 24, 36, 48, and 60 ft.

**P7.2.** Compute the reactions at supports $A$ and $E$ of the three-hinged parabolic arch in Figure P7.2. Next compute the shear, axial load, and moment at points $B$ and $D$, located at the quarter points.

**P7.3.** Determine the axial load, moment, and shear at point $D$ of the three-hinged parabolic arch.

**P7.4.** Determine the reactions at supports $A$ and $C$ of the three-hinged circular arch.

**P7.5.** Compute the support reactions for the arch in Figure P7.5. *(Hint: You will need two moment equations: Consider the entire free body for one, and a free body of the portion of truss to either the left or right of the hinge at $B$).*
P7.6. Determine all bar forces in the three-hinged, trussed arch in Figure P7.6.

P7.7. (a) In Figure P7.7 compute the horizontal reaction $A_x$ at support $A$ for a 10-kip load at joint $B$. (b) Repeat the computation if the 10-kip load is also located at joints $C$ through $F$.

P7.8. For the arch rib to be funicular for the dead loads shown, establish the elevation of the lower chord joints $B$, $C$, and $E$.

P7.9. Determine the reactions at supports $A$ and $E$ of the three-hinged arch in Figure P7.9.

P7.10. Establish the funicular arch for the system of loads in Figure P7.10.

P7.11. If the arch rib $ABCDE$ in Figure P7.11 is to be funicular for the dead loads shown at the top joints, establish the elevation of the lower chord joints at $B$ and $D$. 
P7.12. Computer study of a two-hinged arch. The objective is to establish the difference in response of a parabolic arch to (1) uniformly distributed loads and (2) a single concentrated load.

(a) The arch in Figure P7.12 supports a roadway consisting of simply supported beams connected to the arch by high-strength cables with area $A = 2$ in$^2$ and $E = 26,000$ ksi. (Each cable transmits a dead load from the beams of 36 kips to the arch.) Determine the reactions; the axial force, shear, and moment at each joint of the arch; and the joint displacements. Plot the deflected shape. Represent the arch by a series of straight segments between joints. The arch has a constant cross section with $A = 24$ in$^2$, $I = 2654$ in$^4$, and $E = 29,000$ ksi.

(b) Repeat the analysis of the arch if a single 48-kips vertical load acts downward at joint 18. Again, determine all the forces acting at each joint of the arch, the joint displacements, etc. and compare results with those in (a). Briefly describe the difference in behavior.

P7.13. Computer study of arch with a continuous floor girder. Repeat part (b) in problem P7.12 if a continuous girder with $A = 102.5$ in$^2$ and $I = 40,087$ in$^4$, as shown in Figure P7.13, is provided to support the floor system. For both the girder and the arch, determine all forces acting on the arch joints as well as the joint displacements. Discuss the results of your study of P7.12 and P7.13 with particular emphasis on the magnitude of the forces and displacements produced by the 48-kip load.