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Finite Element Analysis of Fracture in Concrete Structures: State-of-the-Art

Reported by ACI Committee 446

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Fracture is an important mode of deformation and damage in both plain and reinforced concrete structures. To accurately predict fracture behavior, it is often necessary to use finite element analysis. This report describes the state-of-the-art of finite element analysis of fracture in concrete. The two dominant techniques used in finite element modeling of fracture—the discrete and the smeared approaches—are described. Examples of finite element analysis of cracking and fracture of plain and reinforced concrete structures are summarized. While almost all concrete structures crack, some structures are fracture sensitive, while others are not. Therefore, in some instances it is necessary to use a consistent and accurate fracture model in the finite element analysis of a structure. For the most general and predictive finite element analyses, it is desirable to allow cracking to be represented using both the discrete and the smeared approaches.

Keywords: Concrete; crack; cracking; damage; discrete cracking; finite element analysis; fracture; fracture mechanics; reinforced concrete; structures; size effect; smeared cracking.

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CHAPTER 1—INTRODUCTION

In this report, the state-of-the-art in finite element modeling of concrete is viewed from a fracture mechanics perspective. Although finite element methods for modeling fracture are undergoing considerable change, the reader is presented with a snapshot of current thinking and selected literature on the topic.

1.1—Background

As early as the turn of the 19th century, engineers realized that certain aspects of concrete behavior could not be described or predicted based upon classical strength of materials techniques. As the discipline of fracture mechanics has developed over the course of this century (and indeed, is still developing), it has become clear that a correct analysis of many concrete structures must include the ideas of fracture mechanics.

The need to apply fracture mechanics results from the fact that classical mechanics of materials techniques are inadequate to handle cases in which severe discontinuities, such as cracks, exist in a material. For example, in a tension field, the stress at the tip of a crack tends to infinity if the material is assumed to be elastic. Since no material can sustain infinite stress, a region of inelastic behavior must therefore surround the crack tip. Classical techniques cannot, however, handle such complex phenomena. The discipline of fracture mechanics was developed to provide techniques for predicting crack propagation behavior.

Westergaard (1934) appears to have been the first to apply the concepts of fracture mechanics to concrete beams. With the advent of computers in the 1940s, and the subsequent rapid development of the finite element method (FEM) in the 1950s, it did not take long before engineers attempted to analyze concrete structures using the FEM (Clough 1962, Ngo and Scordelis 1967, Nilson 1968, Rashid 1968, Cervenka and Gerstle 1971, Cervenka and Gerstle 1972). However, even with the power of the FEM, engineers faced certain problems in trying to model concrete structures. It became apparent that concrete structures usually do not behave in a way consistent with the assumptions of classical continuum mechanics (Bazant 1976).

Fortunately, the FEM is sufficiently general that it can model continuum mechanical phenomena as well as discrete phenomena (such as cracks and interfaces). Engineers performing finite element analysis of reinforced concrete structures over the past thirty years have gradually begun to recognize the importance of discrete mechanical behavior of concrete. Fracture mechanics may be defined as that set of ideas or concepts that describe the transition from continuous to discrete behavior as separation of a material occurs. The two main approaches used in FEM analysis to represent cracking in concrete structures have been to 1) model cracks discretely (discrete crack approach); and 2) model cracks in a smeared fashion by applying an equivalent theory of continuum mechanics (smeared crack approach). A third approach involves modeling the heterogeneous constituents of concrete at the size scale of the aggregate (discrete particle approach) (Bazant et al. 1990).

Kaplan (1961) seems to have been the first to have performed physical experiments regarding the fracture mechanics of concrete structures. He applied the Griffith (1920) fracture theory (modified in the middle of this century to become the theory of linear elastic fracture mechanics, or LEFM) to evaluate experiments on concrete beams with crack-simulating notches. Kaplan concluded, with some reservations, that the Griffith concept (of a critical potential energy release rate or critical stress intensity factor being a condition for crack propagation) is applicable to concrete. His reservations seem to have been justified, since more recently it has been demonstrated that LEFM is not applicable to typical concrete structures. In 1976, Hillerborg, Modeer and Petersson studied the fracture process zone (FPZ) in front of a crack in a concrete structure, and found that it is long and narrow. This led to the development of the fictitious crack model (FCM) (Hillerborg et al. 1976), which is one of the simplest nonlinear discrete fracture mechanics models applicable to concrete structures.

Finite element analysis was first applied to the cracking of concrete structures by Clough (1962) and Scordelis and his coworkers Nilson and Ngo (Nilson 1967, Ngo and Scordelis 1967, Nilson 1968). Ngo and Scordelis (1967) modeled discrete cracks, as shown in Fig. 1.1, but did not address the problem of crack propagation. Nilson (1967) modeled progressive discrete cracking, not by using fracture mechanics techniques, but rather by using a strength-based criterion. The stress singularity that occurs at the crack tip was not modeled. Thus, since the maximum calculated stress near the tip of a crack depends upon element size, the results were mesh-dependent (nonobjective). Since then, much of the research and development in discrete numerical modeling of fracture of concrete structures has been carried out by Ingraffea and his coworkers (Ingraffea 1977, Ingraffea and Manu 1980, Saouma 1981, Gerstle 1982, Ingraffea 1983, Gerstle 1986, Wawrzynek and Ingraffea 1987, Swenson and Ingraffea 1988, Wawrzynek and Ingraffea 1989, Ingraffea 1990, Martha et al. 1991) and by Hillerborg and coworkers (Hillerborg et al. 1976, Petersson 1981, Gustafsson 1985).

Another important approach to modeling of fracture in concrete structures is called the smeared crack model (Rashid 1968). In the smeared crack model, cracks are modeled by changing the constitutive (stress-strain) relations of the solid continuum in the vicinity of the crack. This approach has been used by many investigators (Cervenka and Gerstle 1972, Darwin and Pecknold 1976, Bazant 1976, Meyer and Bathe 1982, Chen 1982, Balakrishnan and Murray 1988). Bazant (1976) seems to have been the first to realize that, because of its strain-softening nature, concrete cannot be modeled as a pure continuum. Zones of damage tend to localize to a size scale that is of the order of the size of the aggregate.

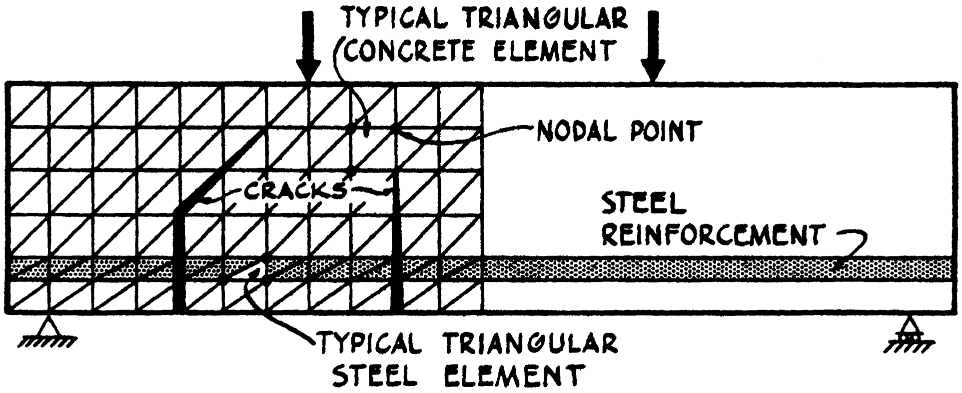

Fig. 1.1—The first finite element model of a cracked reinforced concrete beam (Ngo and Scordelis 1967)

Therefore, for concrete to be modeled as a continuum, account must be taken of the size of the heterogeneous structure of the material. This implies that the maximum size of finite elements used to model strain softening behavior should be linked to the aggregate size. If the scale of the structure is small, this presents no particular problem. However, if the scale of the structure is large compared to the size of its internal structure (aggregate size), stress intensity factors (fundamental parameters in LEFM) may provide a more efficient method for modeling crack propagation than the smeared crack approach (Griffith 1920, Bazant 1976). Most structures of interest are of a size between these two extremes, and controversy currently exists as to which of these approaches (discrete fracture mechanics or smeared cracking continuum mechanics) is more effective. This report describes both the discrete and smeared cracking methods. These two approaches, however, are not mutually exclusive, as shown, for example, by Elices and Planas (1989).

When first used to model concrete structures, it was expected that the FEM could be used to solve many problems for which classical solutions were not available. However, even this powerful numerical tool has proven to be difficult to apply when the strength of a structure or structural element is controlled by cracking. When some of the early finite element analyses are studied critically in light of recent developments, they are found to be nonobjective or incorrect in terms of the current understanding of fracture mechanics, although many produced a close match with experimental results. It is now clear that any lack of success in these models was not due to a weakness in the FEM, but rather due to incorrect approaches used to model cracks. In many cases, success can be achieved only if the principles of fracture mechanics are brought to bear on the problem of cracking in plain and reinforced concrete. These techniques have not only proven to be powerful, but have begun to provide explanations for material behavior and predictions of structural response that have previously been poorly or incorrectly understood.

While some preliminary research has been performed in the finite element modeling of cracking in three-dimensional structures (Gerstle et al. 1987, Wawrzynek and Ingraffea 1987, Ingraffea 1990, Martha et al. 1991), the state-of-the-art in the fracture analysis of concrete structures seems currently to be generally limited to two-dimensional models of structures.

1.2—Scope of report

Several previous state-of-the-art reports and symposium proceedings discuss finite element modeling of concrete structures (ASCE Task Committee 1982, Elfgren 1989, *Computer-Aided* 1984, 1990, *Fracture Mechanics* 1989, Firrao 1990, van Mier et al. 1991, *Concrete Design* 1992, *Fracture* 1992, *Finite Element* 1993, *Computational Modeling* 1994, *Fracture and Damage* 1994, *Fracture* 1995). This report provides an overview of the topic, with emphasis on the application of fracture mechanics techniques. The two most commonly applied approaches to the FEM analysis of fracture in concrete structures are emphasized. The first approach, described in Chapter 2, is the discrete crack model. The second approach, described in Chapter 3, is the smeared crack model. Chapter 4 presents a review of the literature of applications of the finite element technique to problems involving cracking of concrete. Finally, some general conclusions and recommendations for future research are given in Chapter 5.

No attempt is made to summarize all of the literature in the area of FEM modeling of fracture in concrete. There are several thousand references dealing simultaneously with the FEM, fracture, and concrete. An effort is made to crystallize the confusing array of approaches. The most important approaches are described in detail sufficient to enable the reader to develop an overview of the field. References to the literature are provided so that the reader can obtain further details, as desired. The reader is referred to ACI 446.1R for an introduction to the basic concepts of fracture mechanics, with special emphasis on the application of the field to concrete.

CHAPTER 2—DISCRETE CRACK MODELS

A discrete crack model treats a crack as a geometrical entity. In the FEM, unless the crack path is known in advance, discrete cracks are usually modeled by altering the mesh to accommodate propagating cracks. In the past, this remeshing process has been a tedious and difficult job, relegated to the analyst. However, newer software techniques now enable the remeshing process, at least in two-dimensional problems, to be accomplished automatically by the computer. A zone of inelastic material behavior, called the fracture process zone (FPZ), exists at the tip of a discrete crack, in which the two sides of the crack may apply tractions to each other. These tractions are generally thought of as nonlinear functions of the relative displacements between the sides of the crack.

2.1—Historical background

Finite element modeling of discrete cracks in concrete beams was first attempted by Ngo and Scordelis (1967) by introducing cracks into the finite element mesh by separating elements along the crack trajectory, as shown in Fig. 1.1. They did not, however, attempt to model crack propagation. Had they done so, they would have found many problems, starting with the fact that the stresses at the tips of the cracks increase without bound as the element size is reduced, and no convergence (of crack tip stresses) to a solution would have been obtained. Also, in light of the findings of Hillerborg et al. (1976) that a crack in concrete has a gradually softening region of significant length at its tip, it was inaccurate to model cracks with traction-free surfaces. It is notable that Ngo and Scordelis also grappled with the theoretically difficult issue of connecting the reinforcing elements with the concrete elements via "bond-link" elements.

Nilson (1967, 1968) was the first to consider a FEM model to represent *propagation* of discrete cracks in concrete structures. Quoting from his thesis (Nilson 1967):

> The present analysis includes consideration of progressive cracking. The uncracked member is loaded incrementally until previously defined cracking criteria are exceeded at one or more locations in the member. Execution terminates, and the computer output is subjected to visual inspection. If the average value of the principal tensile stress in two adjacent elements exceeds the ultimate tensile strength of the concrete, then a crack is defined between those two elements along their common edge. This is done by establishing two disconnected nodal points at their common corner or corners where there formerly was only one. When the principal tension acts at an angle to the boundaries of the element, then the crack is defined along the side most nearly normal to the principal tension direction.
>
> The newly defined member, with cracks (and perhaps partial bond failure), is then re-loaded from zero in a second loading stage, also incrementally applied to account for the nonlinearities involved. Once again the execution is terminated if cracking criteria are exceeded. The incremental extension of the crack is recorded, and the member loaded incrementally in the third stage, and so on. In this way, crack propagation may be studied and the extent of cracking at any stage of loading is obtained.

The problems associated with this approach to discrete crack propagation analysis are three-fold: (1) cracks in concrete structures of typical size scale develop gradually (Hillerborg et al. 1976), rather than abruptly; (2) the procedure forces the cracks to coincide with the predefined element boundaries; and (3) the energy dissipated upon crack propagation is unlikely to match that in the actual structure, resulting in a spurious solution.

In the 1970s, great strides were made in modeling of LEFM using the FEM. Chan, Tuba, and Wilson (1970) pointed out that a large number of triangular constant stress finite elements are required to obtain accurate stress intensity factor solutions using a displacement correlation technique (about 2000 degrees of freedom are required to obtain 5 percent accuracy in the stress intensity factor solution). At this time, singular finite elements had not yet been developed (singular elements exactly model the stress state at the tip of a crack). In their paper, Chan et al. pointed out that there were then three ways to obtain stress intensity factors from a finite element solution: (1) displacement correlation; (2) stress correlation; and (3) energy release rate methods (line integral or potential energy derivative approaches).

Wilson (1969) appears to have been the first to have developed a singular crack tip element. Shortly thereafter, Tracey (1971) developed a triangular singular crack tip finite element that required far fewer degrees of freedom than analysis with regular elements to obtain accurate stress intensity factor. At about the same time, Tong, Pian, and Lasry (1973) developed and experimented with hybrid singular crack tip elements (including stress-intensity factors, as well as displacement components, as degrees of freedom).

Jordan (1970) noticed that shifting the midside nodes along adjacent sides of an eight-noded quadrilateral toward the corner node by one-quarter of the element's side length caused the Jacobian of transformation to become zero at the corner node of the element. This led to the discovery by Henshell and Shaw (1975) and Barsoum (1976) that the shift allowed the singular stress field to be modeled exactly for an elastic material. Thus, standard quadratic element with midside nodes shifted to the quarter-points can be used as a $r^{-1/2}$ singularity element for modeling stresses at the tip of a crack in a linear elastic medium.

The virtual crack extension method for calculating Mode I stress intensity factors was developed independently by Hellen (1975) and Parks (1974). In this method, G, the rate of change of potential energy per unit crack extension, is calculated by a finite difference approach. This approach does not require the use of singular elements to obtain Mode I (opening mode) stress-intensity factors. Recently, it has been found that by decomposing the displacement field into symmetric and antisymmetric components with respect to the crack tip, the method may also be extended to calculate Mode II (sliding) and Mode III (tearing) energy release rates and stress-intensity factors (Sha and Yang 1990, Shumin and Xing 1990, Rahulkumar 1992).

Having developed the capability to compute stress intensity factors using the FEM, the next big step was to model linear elastic crack *propagation* using fracture mechanics principles. This was started for concrete by Ingraffea (1977), and continued by Ingraffea and Manu (1980), Saouma (1981), Gerstle (1982, 1986), Wawrzynek and Ingraffea (1986), and Swenson and Ingraffea (1988). These attempts were primarily aimed at facilitating the process of discrete crack propagation through automatic crack trajectory computations and semi-automatic remeshing to allow discrete crack propagation to be modeled. Currently, the main technical difficulties involved in modeling of discrete LEFM crack propagation are in the 3D regime. In 2D applications, automatic propagation and remeshing algorithms have been reasonably successful and are improving. In three-dimensional modeling, automatic remeshing algorithms are on the verge of being sufficiently developed to model general crack propagation, and computers are just becoming powerful enough to accurately solve problems with complex geometries caused by the propagation of a number of discrete cracks.

Another development in discrete crack modeling of concrete structures has been the realization that LEFM does not apply to structural members of normal size, because the FPZ in concrete is relatively large compared to size of the member. This has led to the development of finite element modeling of nonlinear discrete fracture—usually as the implementation of the fictitious crack model (FCM) (Hillerborg et al. 1976), in which the crack is considered to be a strain softening zone modeled by cohesive nodal forces or by interface elements [first developed by Goodman, Taylor, and Brekke (1968)].

Finally, there appear to be situations in which even the FCM seems inadequate to model realistic concrete behavior in the FPZ. In this case, a smeared crack model of some kind, as described in Chapter 3, becomes necessary.

2.2—Linear Elastic Fracture Mechanics (LEFM)

Linear elastic fracture mechanics (LEFM) is an important approach to the fracture modeling of concrete structures, even though it is only applicable to very large (say several meters in length) cracks. For cracks that are smaller than this, LEFM over-predicts the load at which the crack will propagate. To determine whether LEFM may be used or whether nonlinear fracture mechanics is necessary for a particular problem, one must determine the size of the steady state fracture process zone (FPZ) compared to the least dimension associated with the crack tip (ACI 446.1R). The FPZ size and the crack tip least dimension are discussed next.

The FPZ may be defined as the area surrounding a crack tip within which inelastic material behavior occurs. The FPZ size grows as load is applied to a crack, until it has developed to the point that the (traction-free) crack begins to propagate. If the size of the FPZ is small compared to other dimensions in the structure, then the assumptions of LEFM lead to the conclusion that the FPZ will exhibit nonchanging characteristics as the crack propagates. This is called the steady state FPZ. The size of the steady state FPZ depends only upon the material properties. In concrete, as opposed to metals, the FPZ can often be thought of as an interface separation phenomenon, with little accompanying volumetric damage. The characteristics of the steady state FPZ depend upon the aggregate size, shape and strength, and upon microstructural details of the particular concrete under consideration. The FPZ was first studied in detail by Hillerborg, Modeer, and Petersson (1976). The size of the FPZ depends on the model used in the study. For example, in the analysis carried out by Ingraffea and Gerstle (1985) for normal strength concrete, the steady state FPZ ranged from 6 in. (150 mm) to 3 ft (1 m) in length.

The least dimension (L.D.) associated with a crack tip is best defined with the aid of Fig. 2.1 (Gerstle and Abdalla 1990). The least dimension is used to calculate an approximate radius surrounding the crack tip within which the singular stress field can be guaranteed to dominate the solution. The least dimension can be defined as the distance from the crack tip to the nearest discontinuity that might cause a local disturbance in the stress field. Fig. 2.1(a) shows the case where the crack tip L.D. is controlled by the proximity to the crack tip of a free surface. Fig. 2.1(b) shows the case where the least dimension is the crack length itself. Fig. 2.1(c) shows the case where the least dimension is controlled by the crack tip passing close by a reinforcing bar. [Of course, if the reinforcement is considered as a smeared (rather than discrete) constituent of the reinforced concrete composite, then it need not be modeled discretely, and the constitutive relations and the FPZ must correspondingly include the effect of the smeared reinforcing bars.] Fig. 2.1(d) shows the case where the least dimension is controlled by the size of the ligament (the remaining uncracked dimension of the member). In Fig. 2.1(e), the least dimension is governed by a kink in the crack. Finally, Fig. 2.1(f) shows an example of the least dimension being controlled by the radius of curvature of the crack.

As explained in Chapter 2 of ACI 446.1R, one of the fundamental assumptions of LEFM is that the size of the FPZ is negligible (say, no more than one percent of the least dimension associated with the crack tip). It is this assumption that allows for a theoretical stress distribution near the crack tip in linear elastic materials, in which the stress varies with $r^{-1/2}$, in which r is the distance from the crack tip. Stress-intensity factors K_I, K_{II}, and K_{III} are defined as the magnitudes of the singular stress fields for Mode I, Mode II, and Mode III cracks, respectively. If the FPZ is not small compared to the least dimension, then singular stress fields may not be assumed to exist, and consequently, K_I, K_{II}, and K_{III} are not defined for such a crack tip. In such a case, the FPZ must be modeled explicitly and a nonlinear fracture model is required.

As mentioned earlier, fracture process zones in concrete can be on the order of 1 ft (0.3 m) or more in length. For the great majority of concrete structures, least dimensions are less than several feet. Therefore, fracture in these types of structures must be modeled using nonlinear fracture mechanics. Only in very large concrete structures, for example, dams, is it possible to apply LEFM appropriately. For dams with large aggregate, possibly on the size scale of meters, LEFM may not be applicable because of the correspondingly larger size of the FPZ.

Even though it is recognized that LEFM is not applicable to typical concrete structures, it is appropriate to review the details of the finite element analysis of LEFM. Then, in Section 2.3, the finite element analysis of nonlinear discrete fracture mechanics will be presented.

2.2.1 *Fracture criteria: K, G, mixed-mode models*

Stress-intensity factors K_I, K_{II}, and K_{III} or energy release rates G_I, G_{II}, and G_{III} may be used in LEFM to predict crack equilibrium conditions and propagation trajectories. There are several theories that can be used to predict the direction of crack propagation. These include, for quasistatic problems, the maximum circumferential tensile stress theory (Erdogan and Sih 1963), the maximum energy release rate theory (Hussain et al. 1974), and the minimum strain energy density theory (Sih 1974). These theories all give practically the same crack trajectories and loads at which crack extension takes place, and therefore the theory of choice depends primarily upon convenience of implementation. Each of these theories may also be applied to dynamic fracture propagation problems (Swenson 1986). As in metals, cyclic fatigue crack propagation in concrete may be modeled with the

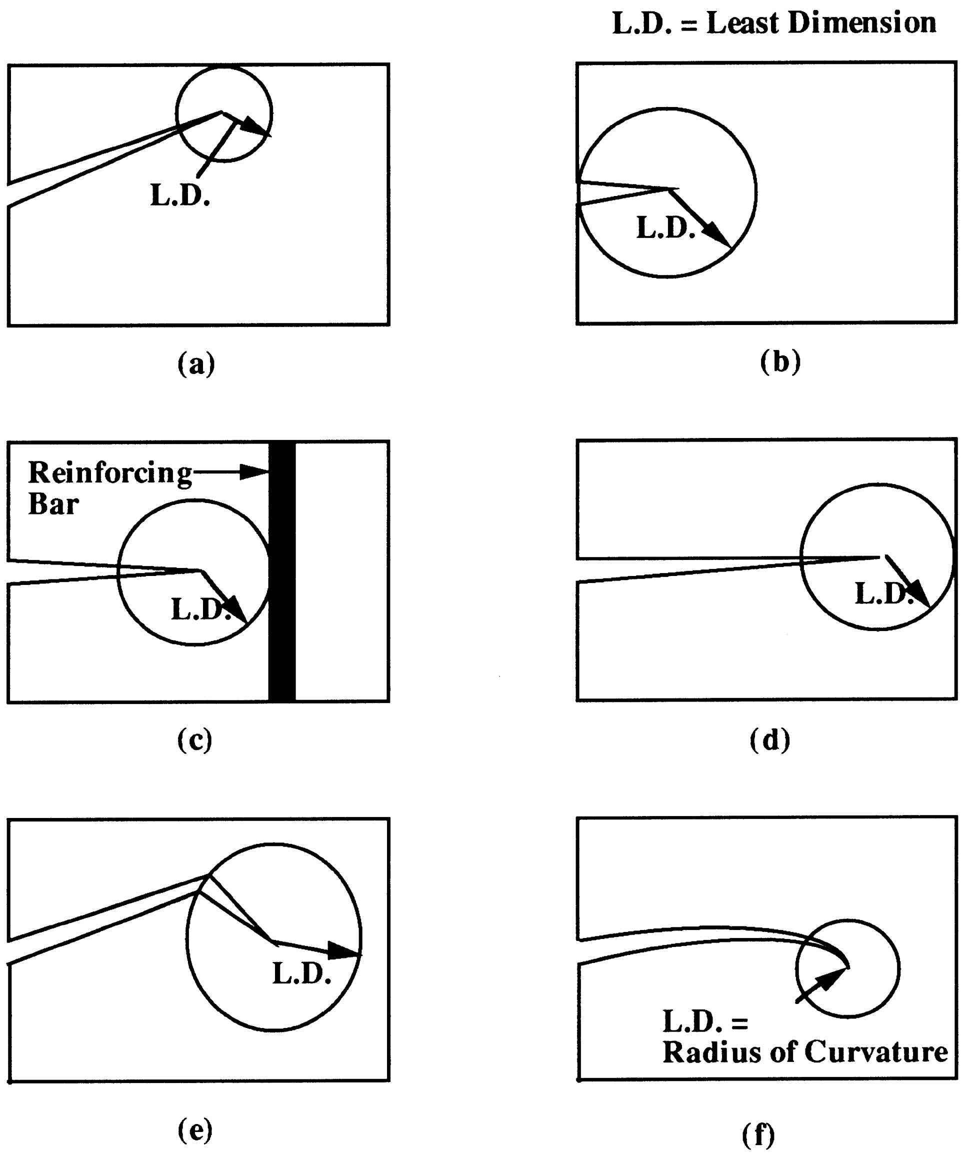

Fig. 2.1—Examples illustrating the concept of "least dimension (L.D.)" associated with a crack tip (Gerstle and Abdalla 1990)

Paris Model (Barsom and Rolfe 1987) in conjunction with the mixed-mode crack propagation theories just mentioned. However, it is rare that an unreinforced concrete structure is both (1) large enough to merit LEFM treatment and (2) subject to fatigue loading.

In most of the literature on discrete crack propagation in concrete structures, it has been considered necessary to model the stress singularity at a crack tip using singular elements. However, accurate results can also be obtained without modeling the stress singularity, but rather by calculating the energy release rates directly (Sha and Yang 1990, Rahulkumar 1992). However, for a comprehensive treatment, we discuss modeling of stress singularities next.

2.2.2 ***FEM modeling of singularities and stress intensity factors***

Special-purpose singular finite elements have been created with stress-intensity factors included explicitly as degrees-of-freedom (Byskov 1970, Tong and Pian 1973, Atluri et al. 1975, Mau and Yang 1977). However, these are special-purpose hybrid elements that are not usually included in standard displacement-based finite element codes, and will not be discussed in further detail here. The most successful displacement-based elements are the Tracey element (Tracey 1971) and the quarterpoint quadratic triangular isoparametric element (Henshell and Shaw 1975, Barsoum 1976, Saouma 1981, Saouma and Schwemmer 1984). Most general purpose finite element codes unfortunately do not include the Tracey element, but they do include six noded triangular elements, which can then be used as singular quarterpoint crack tip elements.

After a finite element analysis has been completed, stress-intensity factors can be extracted by several approaches. The most accurate methods are the energy approaches: the *J*-integral, virtual crack extension, or stiffness derivative methods. However, these approaches are not as easy to apply for the case of mixed-mode crack propagation, and have been applied only rarely to three-dimensional problems (Shivakumar et al. 1988). Simpler to apply (for mixed-mode fracture

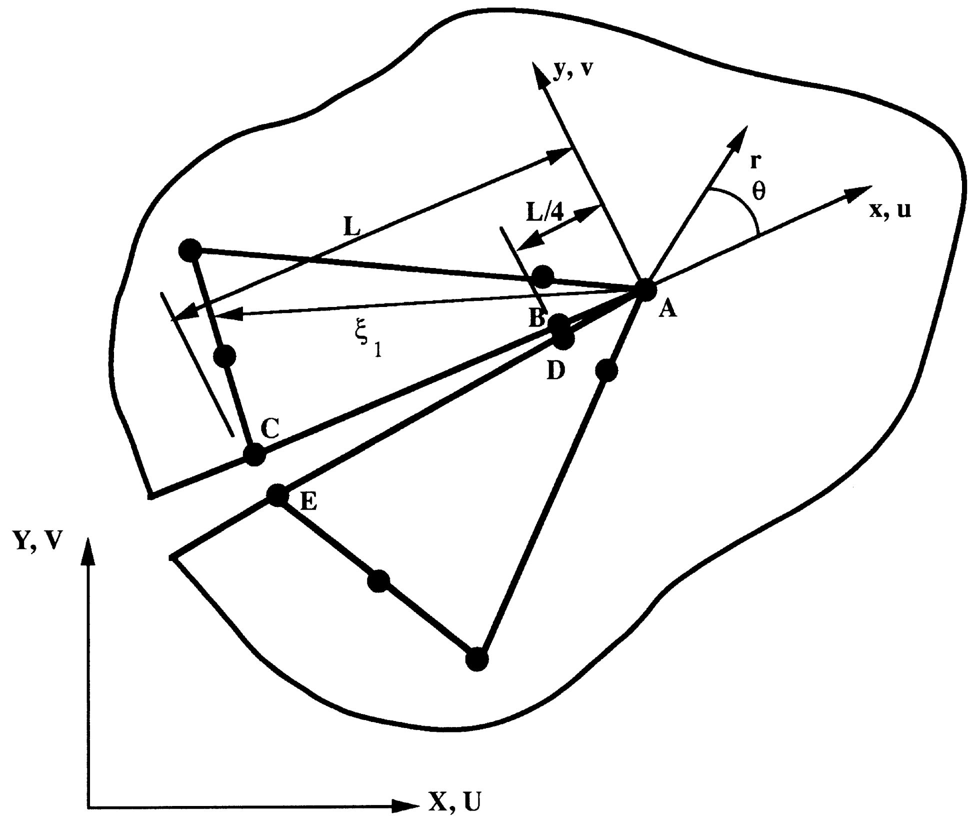

Fig. 2.2—Nomenclature for 2D quarter point singular isoparametric elements

mechanics) are the displacement correlation techniques. Because these techniques sample local displacements at various points, and correlate these with the theoretical displacement field associated with a crack tip, they are generally not as accurate as the energy approaches, which use integrated information. The displacement correlation techniques are usually used only when singular elements are employed, while the energy approaches are used for determining energy release rates for cracks that may or may not be discretized with the help of singular elements.

The displacement and stress correlation techniques assume that the finite element solution near the crack tip is of the same form as the singular near-field solution predicted by LEFM (Broek 1986). By matching the (known) finite element solution with the (known, except for K_I, K_{II}, and K_{III}) theoretical near-field LEFM solution, it is possible to calculate the stress-intensity factors. Since only three equations are needed to obtain the three stress-intensity factors, while many points that can be matched, there are many possible schemes for correlation. These include matching nodal responses only on the crack surfaces and least-squares fitting of all of the nodal responses associated with the singular elements.

The displacement correlation approach is more accurate than the stress correlation approach because displacements converge more rapidly than stresses using the FEM. Therefore only the displacement correlation approach is discussed in detail here (Shih et al. 1976).

Consider a linear elastic isotropic material with Young's modulus E and Poisson's ratio ν. For the case of plane strain, the near-field displacements (u,v), in terms of polar coordinates r and θ, shown in Fig. 2.2, are given by:

$$u = \frac{2K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left[1-2\nu+\sin^2\frac{\theta}{2}\right]+ \tag{2.1}$$

$$\frac{2K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[2-2\nu+\cos^2\frac{\theta}{2}\right]$$

$$v = \frac{2K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[2-2\nu-\cos^2\frac{\theta}{2}\right]+ \tag{2.2}$$

$$\frac{2K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left[-1+2\nu+\sin^2\frac{\theta}{2}\right]$$

in which u and v are parallel and perpendicular to the crack face, respectively.

Now consider a crack tip node surrounded by quarter point triangular elements shown in Fig. 2.2. Interpolating the radial coordinate, r, along the side AC, by using quadratic shape functions associated with nodes A, B, and C, and solving for the natural triangular area coordinate ξ_1 in terms of r, we obtain:

$$\xi_1 = 1-\sqrt{\frac{r}{L}} \tag{2.3}$$

where L is the length of the side AC. Now interpolating the displacements along the side AC by using the computed displacement components at nodes A, B, and C, and using Eq.

(2.3), we obtain the displacements along the crack surface *AC* in terms of *r*. These are given by:

$$u = u_A + (-3u_A + 4u_B - u_C)\sqrt{\frac{r}{L}} + (2u_A - 4u_B + 2u_C)\frac{r}{L} \quad (2.4)$$

$$v = v_A + (-3v_A + 4v_B - v_C)\sqrt{\frac{r}{L}} + (2v_A - 4v_B + 2v_C)\frac{r}{L} \quad (2.5)$$

Similarly the displacements alongside *AE* can be written as:

$$u = u_A + (-3u_A + 4u_D - u_E)\sqrt{\frac{r}{L}} + (2u_A - 4u_D + 2u_E)\frac{r}{L} \quad (2.6)$$

$$v = v_A + (-3v_A + 4v_D - v_E)\sqrt{\frac{r}{L}} + (2v_A - 4v_D + 2v_E)\frac{r}{L} \quad (2.7)$$

Subtracting Eq. 2.6 from 2.4 and subtracting 2.7 from 2.5, the crack opening displacement (*COD*) and crack sliding displacement (*CSD*) are computed as:

$$COD = (4v_B - v_C - 4v_D + v_E)\sqrt{\frac{r}{L}} + (-(4v_B + 2v_C) + 4v_D - 2v_E)\frac{r}{L} \quad (2.8)$$

$$CSD = (4u_B - u_C - 4u_D + u_E)\sqrt{\frac{r}{L}} + (-4u_B + 2u_C + 4u_D - 2u_E)\frac{r}{L} \quad (2.9)$$

Analytical solutions for *COD* and *CSD* can be obtained by evaluating the displacement components *u* and *v* given by Eqs. 2.1 and 2.2 for $\theta = +\pi$ and $\theta = -\pi$ and subtracting the values at $\theta = -\pi$ from the values at $\theta = +\pi$. Equating the like terms in the finite element and the analytical *COD* and *CSD* profiles, the stress intensity factors are given by:

$$K_I = \sqrt{\frac{2\pi}{L}}\frac{E}{2(1+\nu)(3-4\nu)}[4(v_B - v_D) + v_E - v_C] \quad (2.10)$$

$$K_{II} = \sqrt{\frac{2\pi}{L}}\frac{E}{2(1+\nu)(3-4\nu)}[4(u_B - u_D) + u_E - u_C] \quad (2.11)$$

Thus by meshing the crack tip region with quarter-point quadratic triangular elements and solving for the displacements, the stress intensity factors can be computed by using Eqs. 2.10 and 2.11. This technique does not require any special subroutines to develop the stiffness matrix for the singular elements. A single subroutine can be written to calculate the length *L* of the sides *AC* and *AE*, retrieve the displacement components at the nodes *A*, *B*, *C*, *D*, and *E* and thereby compute the stress-intensity factors using Eqs. 2.10 and 2.11.

Ingraffea and Manu (1980) have developed similar equations for the computation of stress-intensity factors in three dimensions with quarterpoint quadratic elements. In three dimensions, the crack tip is replaced by the crack front, the crack edge by the crack face.

Energy approaches for extracting stress-intensity factors make use of the fact that $K_I = [EG_I]^{1/2}$, $K_{II} = [EG_{II}]^{1/2}$, $K_{III} = [EG_{III}/(1 + \nu)]^{1/2}$ for plane stress and $K_I = [EG_I/1 - \nu^2)]^{1/2}$, $K_{II} = [EG_{II}/(1 - \nu^2)]^{1/2}$, $K_{III} = [EG_{III}/(1 + \nu)]^{1/2}$ for plane strain. Here, G_I, G_{II}, and G_{III} are the potential energy release rates created by collinear crack extension due to Mode I, Mode II, and Mode III deformations, respectively. In the simplest approach, the total energy release rate, $G = G_I + G_{II} + G_{III}$ can be calculated by performing an analysis, calculating the total potential energy, π_A, collinearly extending the crack by a small amount ∂a, reperforming the analysis to obtain π_B, and then using a finite difference to approximate *G* as $G = (\pi_A - \pi_B)/\partial a$. If G_I, G_{II}, and G_{III} are required separately, they can be calculated by decomposing the crack tip displacement and the stress fields into Mode I, Mode II, and Mode III components (Rahulkumar 1992).

The stiffness derivative method for determination of the stress-intensity factor for Mode I (2D and 3D) crack problems was introduced by Parks (1974). The method is equivalent to the J-integral approach (described later).

With reference to Fig. 2.3, any set of finite elements that forms a closed path around the crack tip may be chosen. The simplest set to choose is the set of elements around the crack tip.

The stiffness derivative method involves determination of the stress-intensity factor from a calculation of the potential energy decrease per unit crack advance, *G*. For plane strain and unit thickness, the relation between K_I and *G* is

$$G = -\left[\frac{\partial\pi}{\partial a}\right]_{load} = \frac{(1-\nu^2)}{E}K_I^2 \quad (2.12)$$

in which *P* is the potential energy, *a* is the crack length, *E* is Young’s modulus, and ν is Poisson’s ratio.

Parks (1974) shows that the potential energy, π, in the problem is given by:

$$\pi = \frac{1}{2}\{u\}^T[K]\{u\} - \{u\}^T\{f\} \quad (2.13)$$

in which [*K*] is the global stiffness matrix, and {*f*} is the vector of prescribed nodal loads. Eq. 2.13 is differentiated with respect to crack length, *a*, to obtain the energy release rate as

$$\left[\frac{-\partial\pi}{\partial a}\right]_{load} = -\frac{1}{2}\{u\}^T\frac{\partial[K]}{\partial a}\{u\} - \{u\}^T\frac{\partial\{f\}}{\partial a} = K_1^2\frac{(1-\nu^2)}{E} \quad (2.14)$$

The matrix $\frac{\partial[K]}{\partial a}$ represents the change in the structure stiffness matrix per unit of crack length advance. The term $\frac{\partial\{f\}}{\partial a}$ is nil if the crack tip area is unloaded. The key to understanding the stiffness derivative method is to imagine representing an increment of crack advance with the mesh shown in Fig. 2.3 by rigidly translating all nodes on and within a contour Γ_o (see Fig. 2.3) about the crack tip by an infinitesimal amount Δa in the *x*-direction. All nodes on and outside of contour Γ_1 remain in their initial position. Thus the global stiffness matrix [*K*], which depends on only individual element geometries, displacement functions, and elastic material properties, remains unchanged in the regions interior to Γ_o and exterior to Γ_1, and the only contributions to the first term of Eq. 2.14 come from the band of elements between the contours Γ_o and Γ_1. The structure stiffness matrix [*K*] is the sum over all elements of the element stiffness matrices [K_i]. Therefore,

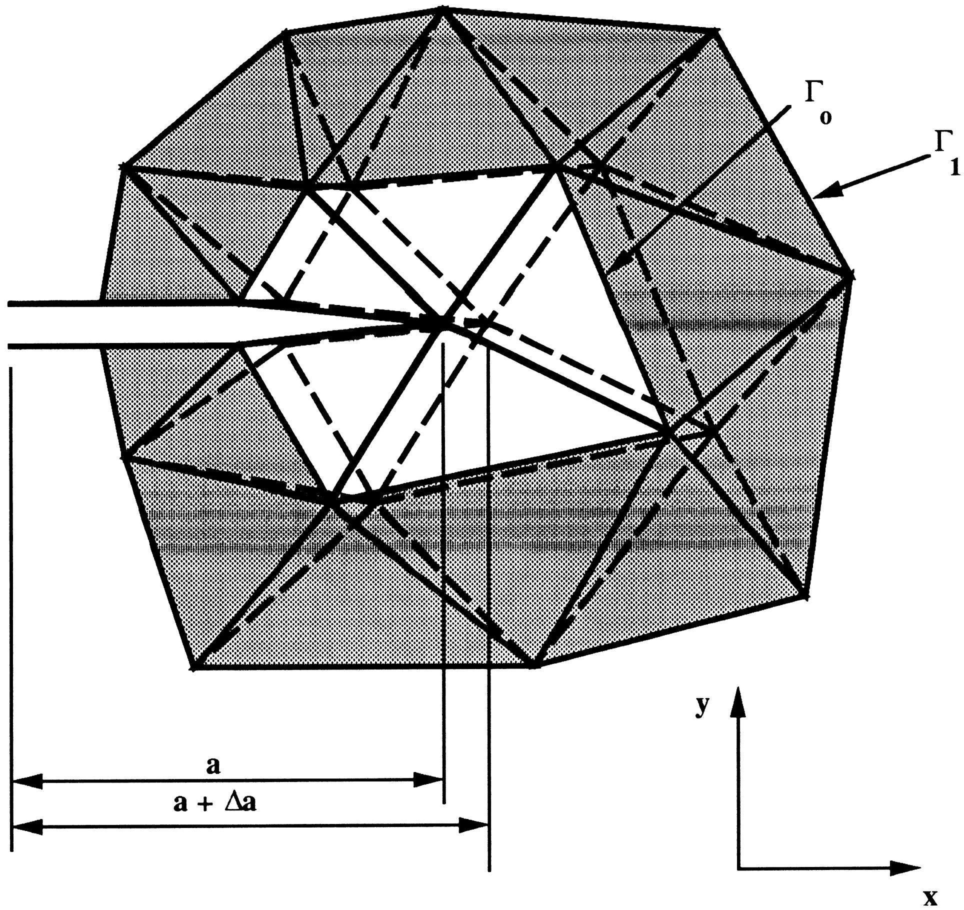

Fig. 2.3—Stiffness derivature approach for advancing nodes (Parks 1978)

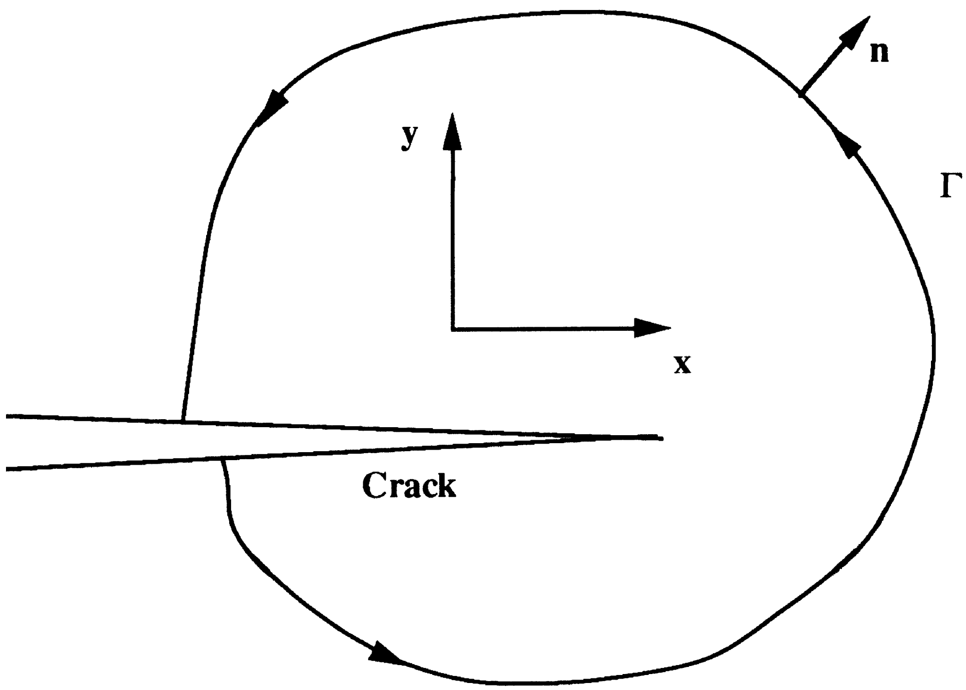

Fig. 2.4—J-Integral nomenclature (Rice 1968)

$$\frac{1}{2}\{u\}^T\frac{\partial[K]}{\partial a}\{u\} = \frac{1}{2}\{u\}^T\sum_{i=1}^{N_c}\frac{\partial[K_i^c]}{\partial a}\{u\} \tag{2.15}$$

in which $[K_i^c]$ is the element stiffness matrix of an element between the contours Γ_o and Γ_1, and N_c is the number of such elements. The derivatives of the element stiffness matrices can be calculated numerically by taking a finite difference:

$$\frac{\partial[K_i^c]}{\partial a} = \frac{\Delta[K_i^c]}{\Delta a} = \frac{1}{\Delta a}[[K_i^c]_{a+\Delta a} - [K_i^c]_a] \tag{2.16}$$

The method may be extended to mixed-mode cracks.

The J-Integral method (Rice 1968) for determining the energy release rate of a Mode I crack is useful for determining energy release rates, not only for LEFM crack propagation, but also for nonlinear fracture problems. For a two-dimensional problem, a path Γ is traversed in a counter-clockwise sense between the two crack surfaces, as shown in Fig. 2.4. The J-integral is defined as:

$$J \equiv \int_\Gamma\left(w\,dx_2 - p_i\frac{\partial u_i}{\partial x_1}ds\right) \tag{2.17}$$

where summation over the range of repeated indices is understood.

Here, $w \equiv \int_0^\varepsilon \sigma_{ij}d\varepsilon_{ij}$, i,j = 1, 2, 3 is the strain energy density, s is the arc length, and p_i is the traction exerted on the body bounded by Γ and the crack surface. The J-integral is equal to the energy release rate G of the crack (Rice 1968).

The J-integral method can be relatively easily applied to a crack problem whose stress and displacement solution is known, and is not limited to linear materials. However, elasticity or pseudoelasticity along the contour, Γ, is a requirement (Rice 1968).

Alternate energy approaches for extraction of stress-intensity factors from three-dimensional problems have been developed (Shivakumar et al. 1988). Bittencourt et al. (1992) provide a single reference that compares the displacement correlation, the J-integral, and the modified crack closure integral techniques for obtaining stress-intensity factors.

When using triangular quarter-point elements to model the singularity at a crack tip, meshing guidelines have been suggested by a number of researchers (Ingraffea 1983, Saouma and Schwemmer 1984, Gerstle and Abdalla 1990). When using the displacement correlation technique to extract stress-intensity factors, the guidelines are summarized as follows:

1. Use a 2 x 2 (reduced) integration scheme (Saouma and Schwemmer 1984).

2. To achieve 5 percent maximum expected error in any stress component due to any mixed-mode problem, use at least eight approximately equiangular singular elements adjacent to the crack tip node. For 1 percent error, 16 singular elements should be used (Gerstle and Abdalla 1990).

3. There is an optimal size for the crack tip elements. If they are too small, they do not encompass the near-field region of the solution, and surrounding regular elements will be "wasted" modeling the near field. If they are too big, they do not model the far-field solution accurately. The singular elements should be related to the size of the region within which near-field solution is valid. For 5 percent accuracy in stress-intensity factors, the singular elements should be about $^1/_5$ of the size of the least dimension associated with the crack tip. For one percent accuracy, the singular elements should be about $^1/_{20}$ of the size of the least dimension associated with the crack tip (Gerstle and Abdalla 1990).

4. Regular quadratic elements should be limited in size, s, by their clear distance, b, from a crack tip. The ratio of s/b should not exceed unity to achieve 30 percent error, and should not exceed 0.2 to achieve 1 percent error in the near field solution (Gerstle and Abdalla 1990).

The meshing criteria given above show that a large number of elements are required at a crack tip to obtain accurate near-field stresses. Experience shows that 300 degrees of freedom are required per crack tip to reliably obtain 5 percent accuracy in the near field stresses (Gerstle and Abdalla 1990). This becomes prohibitive from a computational standpoint for problems with more than one crack tip.

Fortunately, it is not necessary to accurately model near-field stresses to calculate accurate stress intensity factors. In fact, using no singular elements, energy methods can be used, as described above, to obtain accurate stress intensity factors with far fewer than 300 degrees of freedom per crack tip.

2.3—Fictitious Crack Model (FCM)

Since 1961, there has been a growing realization that LEFM is not applicable to concrete structures of normal size and material properties (Kaplan 1961, Kesler et al. 1972, Bazant 1976). The FPZ ranges from a few hundred millimeters to meters in length, depending upon how the FPZ is defined and upon the properties of the particular concrete being considered (Hillerborg et al. 1976; Ingraffea and Gerstle 1985; Jenq and Shah 1985). The width of the FPZ is small compared to its length (Petersson 1981). LEFM, although not applicable to small structures, may still be applicable to large structures such as dams (Elfgren 1989). However, even for very large structures, when mixed-mode cracking is present the FPZ may extend over many meters; this is due to shear and compressive normal forces (tractions) caused by friction, interference, and dilatation (expansion) between the sides of the crack, far behind the tip of the FPZ. To clarify this notion, Gerstle and Xie (1992) have used an "interface process zone (IPZ)" to model the FPZ.

The fictitious crack model (FCM) has become popular for modeling fracture in concrete (Hillerborg et al. 1976, Petersson 1981, Ingraffea and Saouma 1984, Ingraffea and Gerstle 1985, Gustafsson 1985, Gerstle and Xie 1992, Feenstra et al. 1991a, 1991b, Bocca et al., 1991, Yamaguchi and Chen 1991, Klisinski et al. 1991, Planas and Elices 1992, 1993a, 1993b). Fig. 2.5 shows the terminology and concepts associated with the FCM. This model assumes that the FPZ is long and infinitesimally narrow. The FPZ is characterized by a "normal stress versus crack opening displacement curve," which is considered a material property, as shown in Fig. 2.5.

The FCM assumes that the FPZ is collapsed into a line in 2D or a surface in 3D. A natural way to incorporate the model into the finite element analysis is by employing interface elements. The first interface element was formulated by

Goodman et al. (1968) and was used in the modeling of rock joints. Since then, many types of interface and thin layer elements have been developed and are widely used in geotechnical engineering (Heuze and Barbour 1982, Desai et al. 1984). Zero-thickness elements are the most widely used type of interface, with normal and shear stresses and relative displacements across the interface as constitutive variables. Unrealistic jumps in the results of adjacent integration points of contiguous interfaces have been reported by some authors depending on initial stiffness and load conditions, although most of these problems seem to disappear with the appropriate selection of integration points and integration rule (Gens et al. 1988, Hohberg 1990, Rots and Schellenkens 1990, Schellenkens and De Borst 1993). Other investigators have implemented a semi-discrete FCM by including strain discontinuities (Ortiz et al. 1987, Fish and Belytschko 1988, Belytschko et al. 1988, Dahlblom and Ottosen 1990, Klisinski et al. 1991) or displacement discontinuities (Dvorkin et al. 1990, Lotfi 1992, Lotfi and Shing 1994, 1995) within continuum elements.

The FCM has been incorporated into finite element codes with the use of interface elements. Ingraffea and coworkers (Ingraffea et al. 1984, Ingraffea and Saouma 1984, Ingraffea and Gerstle 1985, Bittencourt, Ingraffea and Llorca 1992) extended the FCM to simulate mixed-mode crack propagation analysis employing six-noded interface elements. Swenson and Ingraffea (1988) used six-noded interface elements to model mixed-mode dynamic crack propagation. Bocca, Carpinteri, and Valente (1991) have published similar work. Gerstle and Xie (1992) used a simple four-noded linear dis-

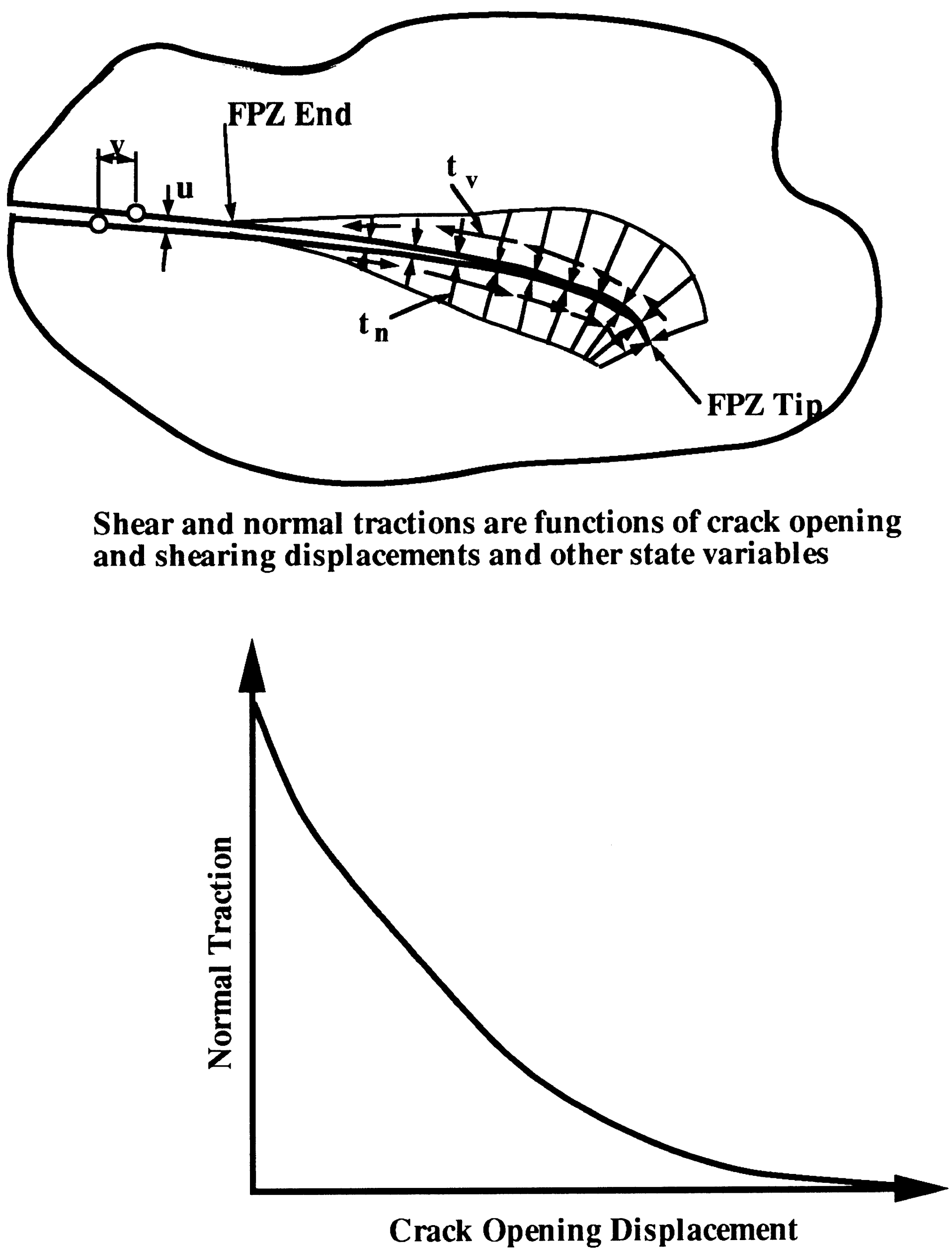

Fig. 2.5—Terminology and concepts associated with the fictitious crack model (FCM) (Hillerborg et al. 1976)

placement interface element that was modified to allow an arbitrary distribution of tractions along its length.

Other references that implement the FCM include Rots (1988), Stankowski (1990), Stankowski et al. (1992), Hohberg (1992a), Lotfi (1992), Vonk (1992), Lotfi and Shing (1994), Garcia-Alvarez et al. (1994), Lopez and Carol (1995), and Bazant and Li (1995).

In the FCM, the stiffness of the interface element is a nonlinear function of the crack opening displacement, so that a nonlinear solution procedure is required. As with any other kind of nonlinear constitutive relation, FEM calculations with interface elements behaving in accordance the FCM require a nonlinear solution strategy. The various existing techniques such as classic Newton iteration, dynamic relaxation, and arc-length procedures have been used, with satisfactory results reported in the literature (Swenson and Ingraffea 1988, Gerstle and Xie 1992, Papadrakis 1981, Underwood 1983, Bathe 1982.)

When using interface elements to model the FPZ, the elements must be very stiff prior to crack initiation to represent an uncracked material (i.e., to keep the two sides of the potential crack together). However, care must be taken not to use a stiffness so high as to cause nonconvergent numerical behavior in the finite element solution. Brown et al. (1993) successfully used interface elements with an axial stiffness equal to 50 times the stiffness of the adjacent concrete elements without numerical difficulties. Gerstle and Xie (1992) suggested using a precrack stiffness equal to the secant stiffness to a point on the descending normal traction-*COD* curve (Fig. 2.5) equal to a *COD* of $^1/_{20}$ to $^1/_{30}$ of the *COD* at which the normal traction drops to zero.

Some investigators have dispensed with interface elements and have instead simulated the FPZ using an influence function approach (Li and Liang 1986, Planas and Elices 1991) in which cohesive forces are applied to the crack faces (Gopalaratnam and Ye 1991). Weighted multipliers are used in the superposition of FEM solutions to satisfy overall equilibrium, compatibility and stress-crack width relations within the FPZ. This approach results in the solution of a set of nonlinear algebraic equations to determine the multipliers. In some cases, it may be appropriate to linearize the relationship between the *COD* and the tractions on the FPZ. Then linear equations can be solved to obtain the solution efficiently (Gopalaratnam and Ye 1991, Li and Bazant 1994).

Extension of the FCM with interface elements to mixed mode cracking requires a constitutive relation for the interface, in which normal and shear stresses and relative displacements are fully coupled. Crack opening and closing conditions are expressed with a biaxial failure surface in the normal-shear stress space. Crack surface displacements, thus, have two components: opening and sliding. Several models of this kind have been proposed recently, all based on the framework of non-associated work hardening plasticity, to obtain a formulation that is fully consistent and contains fracture energy parameters (Stankowski 1990, Stankowski at al. 1993, Lotfi 1992, Lotfi and Shing 1994, Hohberg 1992a, 1992b, Vonk 1992, Garcia-Alvarez et al. 1994). After the crack is completely open, the models prevent interpenetration and provide Coulomb-type friction between crack surfaces. None of the models, however, provides secant unloading in pure tension, as is usual in the classic FCM, because this would complicate the model considerably, as discussed by Carol and Willam (1994).

An approach to nonlinear mixed-mode discrete crack propagation analysis was proposed by Ingraffea and Gerstle (1985). In this approach, the FPZ is modeled by interface elements, with the singular elements used in LEFM placed around the fictitious crack tip to predict the direction of the crack propagation. However, singular elements are not necessary for this purpose if energy release rates are calculated directly (Rice 1968, Parks 1974, Sha and Yang 1990, Rahulkumar 1992).

Another potential approach to determining the direction of a crack is based on the very reasonable assumption that the crack will propagate when the maximum tensile principal stress at the crack tip reaches the strength of the material (Petersson 1981, Gustafsson 1985, Hillerborg and Rots 1989, Bocca et al. 1991, Gerstle and Xie 1992). The direction of crack propagation is assumed to be perpendicular to the maximum tensile principal stress. The problem with this approach is that when the FPZ becomes small compared to the crack tip element size, objectivity of the results is lost. Therefore it makes more sense to use an energy-based approach to determine crack propagation. A basis for such an approach has been developed by Li and Liang (1992). Promising results have been obtained by using energy release rate approaches to determine both the direction and the load level at which a fictitious crack will propagate (Xie et al. 1995).

2.4—Automatic remeshing algorithms

In 1981, work was completed on a two-dimensional fracture propagation code that used simple interactive computer graphics to interactively model crack propagation (Saouma 1981). Many of the tasks that Ingraffea (1977) had performed by editing files manually were now performed interactively. These tasks included semi-automatic remeshing to allow the crack to advance, limited post-processing to view stresses, deformed mesh, and stress-intensity factors, and predictions based upon the mixed-mode crack propagation theories of the crack trajectory.

Subsequently, Wawrzynek and Ingraffea (1987) developed a second generation two-dimensional interactive graphical finite element fracture simulation code based upon a winged-edge topological data structure. This program demonstrated the value of using a topological data structure in fracture simulation codes. More recently, Gerstle and Xie (1992), in collaboration with others, developed an interactive graphical finite element code that is capable of representing and automatically propagating cracks in two dimensional problems.

Procedures have also been developed to handle numerical discretization and arbitrary fracture simulation in three dimensions (Martha 1989, Sousa et al. 1989, Ingraffea 1990, Martha et al. 1991).

A number of algorithms have been introduced for automatic meshing of solid models (Shepard 1984, Cavendish et al. 1985, Schroeder and Shepard 1988, Perucchio et al. 1989). These algorithms can be categorized into three broad families: element extraction, domain triangulation, and recursive spatial decomposition. Although substantial differences exist between these families, the algorithms involve the development of a geometric representation of the structure, which provides the basis for the construction of the finite element mesh (Sapadis and Perucchio 1989). Automatic modeling of discrete crack propagation in three dimensions remains a challenge.

It is worth noting that remeshing may not be required 1) if the crack path is known in advance due to symmetry or due to previous experimental or analytical experience with the same geometry; or 2) if interface elements are placed along all possible crack paths.

CHAPTER 3—SMEARED CRACK MODELS

Early in the application of finite element analysis to concrete structures (Rashid 1968), it became clear that it is often much more convenient to represent cracks by changing the constitutive properties of the finite elements than to change the topography of the finite element grid. The earliest procedure involved dropping the material stiffness to zero in the direction of the principal tensile stress, once the stress was calculated as exceeding the tensile capacity of the concrete. Simultaneously, the stresses in the concrete were released and reapplied to the structure as residual loads. Models of this type exhibit a system of distributed or “smeared” cracks. Ideally, smeared crack models should be capable of representing the propagation of a single crack, as well as a system of distributed cracks, with reasonable accuracy.

Over the years, a number of numerical and practical problems have surfaced with the application of smeared crack models. Principal among these involve the phenomenon of “strain localization.” When microcracks form, they often tend to grow nonuniformly into a narrow band (called a “crack”). Under these conditions, deformation is concentrated in a narrow band, while the rest of the structure experiences much smaller strains. Because the band of localized strain may be so narrow that conventional continuum mechanics no longer applies, various “localization limiters” have been developed. These localization limiters are designed to deal with problems associated with crack localization and spurious mesh sensitivity that are inherent to softening models in general and smeared cracking in particular. Critical reviews of the practical aspects of smeared crack models are presented by ASCE Task Committee (1982) and Darwin (1993).

3.1—Reasons for using smeared crack models

The smeared crack approach, introduced by Rashid (1968), has become the most widely used approach in practice. Three reasons may be given for adopting this approach:

1. The procedure is computationally convenient.

2. Distributed damage in general and densely distributed parallel cracks in particular are often observed in structures (measurements of the locations of sound emission sources provide evidence of a zone of distributed damage in front of a fracture).

3. At many size scales, a crack in concrete is not straight but highly tortuous, and such a crack may be adequately represented by a smeared crack band.

There are, however, serious problems with the classical smeared crack models. They are in principle nonobjective, since they can exhibit spurious mesh sensitivity (Bazant 1976), i.e., the results may depend significantly on the choice of the mesh size (element size) by the analyst. For example, in a tensioned rectangular plain concrete panel with a rectangular finite element mesh, the cracking localizes into a one element wide band. The crack band becomes narrower and increases in length as the mesh is refined, as shown in Fig. 3.1 (Bazant and Cedolin 1979, 1980, Bazant and Oh 1983, Rots et al. 1984, Darwin 1985). Consequently, if not accounted for in the crack model, the load needed to extend the crack band into the next element is less for a finer mesh (for a crack model controlled by tensile strength alone, it decreases roughly by a factor of $\sqrt{2}$ if the element size is halved, and tends to zero as the element size tends to zero). Thus, the maximum load decreases as the mesh is refined (Bazant and Cedolin 1980). Furthermore, the apparent energy that is consumed (and dissipated) during structural failure depends on the mesh size, and tends to zero as the mesh size tends to zero. Such behavior, which is encountered not only for cracking with a sudden stress drop but also for gradual crack formation with a finite slope of the post-peak tensile strain-softening stress-strain diagram that is independent of element size (Bazant and Oh 1983), is nonobjective. These problems make the classical smeared crack approach unacceptable, although in some structures such lack of objectivity might be mild or even negligible [this latter behavior occurs especially when the failure is controlled by yielding of reinforcement rather than cracking of concrete (Dodds, Darwin, and Leibengood 1984)].

To avoid the unobjectivity or spurious mesh sensitivity, a mathematical device called a “localization limiter” must be introduced.

3.2—Types of localization limiters

Several types of localization limiters have been proposed:

3.2.1. ***Crack band model***

The simplest localization limiter is a relationship between the element size and the constitutive model so that the total energy dissipated will match that of the material being modeled. This can be done by adjusting the downward slope of the stress-strain curve, or, equivalently, the value of ε_{max}, shown in Fig. 3.1, as the element size is altered. ε_{max} is increased as the element size is decreased. This procedure, known as the crack band model, has a limitation for coarse meshes (large elements)—ε_{max} cannot be conveniently reduced below the value of strain corresponding to the peak stress, σ_{max}.

More advanced constitutive models based on the theories of plasticity and damage may not exhibit the simple stress-strain relationship shown in Fig. 3.1. Crack band modes can still be applied as long as a local fracture energy is included among the model’s parameters (Pramono and Willam 1987, Carol et al. 1993).

Practically, the most important feature of the crack band model is that it can represent the effect of the structure size on: 1) the maximum capacity of the structure (Bazant 1984); and 2) the slope of the post-peak load-deflection diagram.

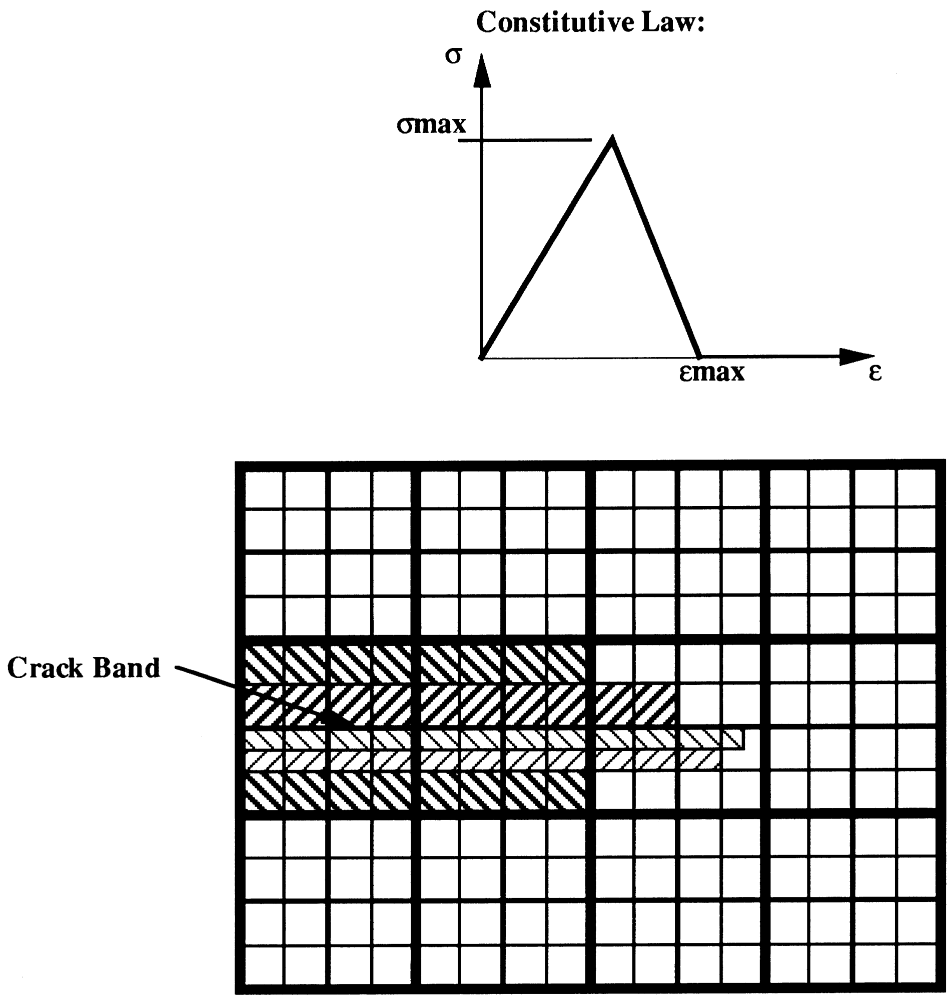

Fig. 3.1—If the constitutive model is independent of element size, the crack band becomes longer and narrower as the mesh is refined

However, from the physical viewpoint, the width of the cracking zone at the front of a continuous fracture (i.e., the FPZ) is represented by a single, element-wide band and cannot be subdivided further; consequently, possible variations in the process zone size, which cause variation of effective fracture energy (i.e., R-curve behavior) cannot be captured and the stress and strain states throughout the FPZ cannot be resolved. The procedure, however, has been widely and successfully applied.

3.2.2. ***Nonlocal continuum***

a) Phenomenological Approach

A general localization limiter is provided by the nonlocal continuum concept and spatial averaging (Bazant 1986).

A nonlocal continuum is a continuum in which some field variables are subjected to spatial averaging over a finite neighborhood of a point. For example, average (nonlocal) strain is defined as

$$\bar{\varepsilon}(x) = \frac{1}{V_r(x)}\int_V \alpha(x-s)\varepsilon(s)dV(s) = \int_V \alpha'(x,s)\varepsilon(s)dV(s) \qquad (3.1)$$

in which $V_r(x) = \int_V \alpha(x-s)dV(s)$ and $\alpha'(x,s) = \alpha(x-s)/(V_r x)$; $\varepsilon(x)$ is the strain at the point in space defined by coordinate vector x; V is the volume of the structure; V_r is the representative volume of the material, as shown in Fig. 3.2, understood to be the smallest volume for which the heterogeneous material can be treated as a continuum (the size of V_r is determined by a characteristic length, L, which is a material property; α is a weighting function, which decays with distance from point x and is zero or nearly zero at points sufficiently remote from x; and the superimposed bar denotes the averaging operator. The dummy variable s represents the spatial coordinate vector in the integral.

As the simplest form of the weighting function, one may consider $\alpha = 1$ within a certain representative volume V_0 centered at point x and $\alpha = 0$ outside this volume. Convergence of numerical solutions, however, is better if α is a smooth bell-shaped function. An effective choice is

$$\alpha = \left[1-\left(\frac{r}{\rho_o L}\right)^2\right]^2 \text{ if } |r| < \rho_o L\text{ ; } \alpha = 0 \text{ if } |r| > \rho_o L \qquad (3.2)$$

in which $r = |x\text{-}s|$ = distance from point x, L = characteristic length (material property), and ρ_o = coefficient chosen in such a manner that the volume under function α given by Eq. 3.2 is equal to the volume under the function $\alpha = 1$ for $r < L/2$ and $\alpha = 0$ for $r > L/2$ (which represents a line segment in

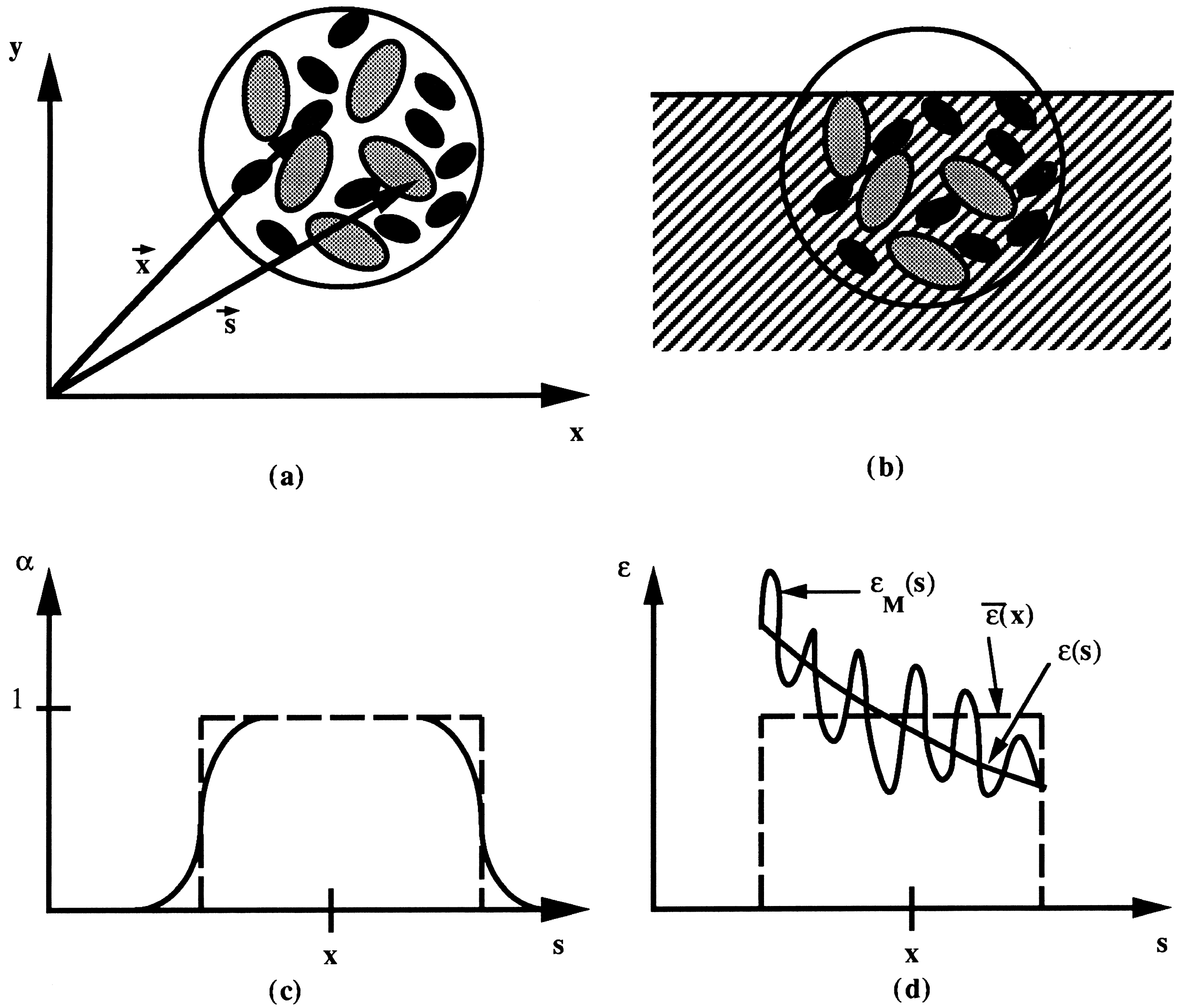

Fig. 3.2—Heterogeneity of concrete at the size scale of the aggregate

1D, a circle in 2D, and a sphere in 3D). Alternatively, the normal distribution function has been used in place of Eq. 3.2 and found to work well enough, although its values are nowhere exactly zero (Bazant 1986).

For points whose distance from all the boundaries is larger than $\rho_o L$, $V_r(x)$ is constant; otherwise the averaging volume protrudes outside the body, and $V_r(x)$ must be calculated for each point to account for the locally unique averaging domain (Fig. 3.2b).

In finite element computations, the spatial averaging integrals are evaluated by finite sums over all integration points of all finite elements of the structure. For this purpose, the matrix of the values of α' for all integration points is computed and stored in advance of the finite element analysis.

This approach makes it possible to refine the mesh as required by structural considerations. Since the representative volume over which structural averaging takes place is treated as a material property, convergence to an exact continuum solution becomes meaningful and the stress and strain distributions throughout the FPZ can be resolved.

The nonlocal continuum model for strain-softening of Bazant et al. (1984) involves the nonlocal (averaged) strain $\bar{\varepsilon}$ as the basic kinematic variable. This corresponds to a system of imbricated (i.e., overlapping in a regular manner, like roof tiles) finite elements, overlaid by a regular finite element system. Although this imbricate model limits localization of strain softening and guarantees mesh insensitivity, the programming is complicated, due to the nonstandard form of the differential equations of equilibrium and boundary conditions, i.e., energy considerations involve the nonlocal strain $\bar{\varepsilon}$.

These problems led to the idea of a partially nonlocal continuum in which stress is based on nonlocal strain, but local strains are retained. Such a nonlocal model, called "the nonlocal continuum with local strain" (Bazant, Pan, and Pijaudier-Cabot 1987, Bazant and Lin 1988, Bazant and Pijaudier-Cabot 1988, 1989) is easier to apply in finite element programming. In this formulation, the usual constitutive relation for strain softening is simply modified so that all of the state variables that characterize strain softening are calculated from nonlocal rather than local strains. Then, all that is necessary to change in a local finite element program is to provide a subroutine that delivers (at each integration point of each element, and in each iteration of each loading step) the value of $\bar{\varepsilon}$ for use in the constitutive model.

Practically speaking, the most important feature of a nonlocal finite element model is that it can correctly represent the effect of structure size on the ultimate capacity, as well as on the post-peak slope of the load-deflection diagram. Nonlocal models can also offer an advantage in the overall speed of solution (Bazant and Lin 1988). Although the numerical effort is higher for each iteration when using a nonlocal model, the formulation lends a stabilizing effect to the solution, allowing convergence in fewer iterations.

b) Micromechanical Approach

Another nonlocal model for solids with interacting microcracks, in which nonlocality is introduced on the basis of microcrack interactions, was developed by Bazant and Jirasek (Bazant 1994, Jirasek and Bazant 1994, Bazant and Jirasek 1994) and applied to the analysis of size effect and localization of cracking damage (Jirasek and Bazant 1994). The model represents a system of interacting cracks using an integral equation that, unlike the phenomenological nonlocal model, involves a spatial integral that represents microcrack interaction based on fracture mechanics concepts. At long range, the integral weighting function decays with the square or cube of the distance in two or three dimensions, respectively. This model, combined with a microplane model, has provided consistent results in the finite element analysis of fracture and structural test specimens (Ozbolt and Bazant 1995).

3.2.3. ***Gradient models***

Another general way to introduce a localization limiter is to use a constitutive relation in which the stress is a function of not only the strain but also the first or second spatial derivatives (or gradients) of strain. This idea appeared originally in the theory of elasticity. A special form of this idea, called Cosserat continuum and characterized by the presence of couple stresses, was introduced by Cosserat and Cosserat (1909) as a continuum approximation of the behavior of crystal lattices on a small scale. A generalization of Cosserat continuum, involving rotations of material points, is the micropolar continuum of Eringen (1965, 1966).

Bazant et al. (1984) and Schreyer (1990) have pointed out that spatial gradients of strains or strain-related variables can serve as a localization limiters. It has been shown (Bazant et al. 1984) that expansion of the averaging integral (Eq. 3.1) into a Taylor series generally yields a constitutive relation in which the stress depends on the second spatial derivatives of strain, if the averaging domain is symmetric and does not protrude outside the body, and on the first spatial derivatives (gradients) of strain, if this domain is unsymmetric or protrudes outside the domain, as happens for points near the boundary (Fig. 3.2b). Since the dimension of a gradient is length^{-1} times that of the differentiated variable, introduction of a gradient into the constitutive equation inevitably requires a characteristic length, L, as a material property. Thus, the use of spatial gradients can be regarded as an approximation to nonlocal continuum, or as a special case.

In material research of concrete, the idea of a spatial gradient appeared in the work of L'Hermite et al. (1952), who found that, to describe differences between observations on small and large specimens, the formation of shrinkage cracks needs to be assumed to depend not only on the shrinkage stress but also on its spatial gradient [see also L'Hermite et al. (1952) and a discussion by Bazant and Lin (1988)]. This was probably the first appearance of the nonlocal concept in fracture. The idea that the stress gradient (or equivalently the strain gradient) influences the material response has also been demonstrated by Sturman, Shah, and Winter (1965) and by Karsan and Jirsa (1969) for combined flexure and axial loading.

The nonlocal averaging integral is meaningful only if the finite elements are not larger than about one third of the representative volume, V_r (Droz and Bazant 1989). The gradient approach, on the other hand, offers the possibility of using finite elements with volumes as large as the representative volume, V_r. Thus, the gradient approach offers the possibility of using a smaller number of finite elements in the analysis. It appears, however, that the programming may be more complicated and less versatile than for the spatial averaging integrals. The problem is that interelement continuity must be enforced not only for the displacements but also for the strains. This requires the use of higher-order elements, or alternatively, if first-order elements are preferred, the use of an independent strain field with separate first-order finite elements.

Nonlocal constitutive models for concrete are reviewed in ACI 446.1R.

CHAPTER 4—LITERATURE REVIEW OF FEM FRACTURE MECHANICS ANALYSES

4.1—General

To help provide an overview of the state-of-the-art in finite element modeling of plain and reinforced concrete structures, a number of representative analyses are summarized in this chapter. Emphasis is placed on easily available, published analyses that attempt to address the fracture behavior of concrete structures. Wherever possible, problems solved using the discrete cracking approach are compared to solutions using the smeared crack approach. Symposium proceedings (Mehlhorn et al. 1978, *Computer-Aided* 1984, 1990, Firrao 1990, *Fracture Mechanics* 1989, van Mier et al. 1991, *Concrete Design* 1992, *Size Effect* 1994, *Computational Modeling* 1994, *Fracture and Damage* 1994, *Fracture Mechanics* 1995) provide the readers with the changing flavor of the state-of-the-art over the years. Many more published FEM fracture mechanics analyses exist than are presented in this chapter; however, it is fair to say that those referenced here are representative of the state-of-the-art.

The term "size effect," used throughout this chapter, is a term that has taken on a special meaning for quasibrittle materials, such as concrete and rock. It describes the *decrease in average stress at failure with increasing member size* that is directly attributable to the well-established fact that fracture is governed by a fracture parameter(s) that depends on the dimensions of the crack (which is tied to structure size) as well as some measure of stress. For concrete, it is not clear if the fracture parameter that governs failure is a material property or if it also depends on the structure size. However, this last point does not alter the general meaning of the term "size effect." In contrast, in the field of fracture of metals, the term “size effect” is understood to apply only to the dependence of fracture parameter (not the average stress) on the structural dimensions.

4.2—Plain concrete

Unreinforced concrete structures are the most fracture sensitive. While usually reinforced with steel in design practice to provide adequate tensile strength, many structures (or parts of structures), for one reason or another, are unreinforced. It makes sense to study the analysis of unreinforced concrete structures, because these provide the most severe tests of fracture behavior, and because the results of these analyses can add insight into the more complex behavior evidenced in reinforced concrete structures. In what follows, short descriptions (listed in chronological order) of finite element analyses of unreinforced concrete structures are presented.

4.2.1 ***Tensile failure***

The plain concrete uniaxial tension specimen is probably more sensitive to fracture than any other type. For this reason, it provides a good test of the fracture-sensitivity of a finite element model for concrete. Although a number of such analyses have been reported, the following references are representative of first a discrete, then a local smeared, then a nonlocal smeared, and finally a micromechanics (random particle) approach.

Gustaffson (1985) used the fictitious crack model (discrete crack) to study the uniaxial tension specimen tested by Petersson (1981). Except for the fictitious crack, he assumed the continuum was linear elastic. He included the influence of initial stresses in his model. Good agreement with test results was obtained. Gustaffson (1985) also studied a number of other specimen geometries using the finite element method. Only Mode I cracks, with trajectories known in advance, were modeled. Interestingly, for larger specimens this analysis approach would still be expected to yield fracture-correct results (as long as a sufficient number of degrees of freedom are used to accurately represent the distribution of tractions along the FPZ). This type of model is extremely powerful for the class of problems in which the crack trajectory is known in advance, and the failure mechanism involves a single discrete crack.

Bazant and Prat (1988a, 1988b) developed and used a (smeared crack) microplane model for brittle-plastic materials. The results were compared with Petersson’s experimental uniaxial tension results (Petersson 1981), along with eight other series. The term microplane model is used to describe a class of constitutive models in which the material behavior is specified independently for planes of many orientations (called microplanes); the macroscopic stresses and strains are obtained through the use of a suitable superposition technique, such as the principle of virtual work. In the study by Bazant and Prat, the agreement between the numerical and experimental results was good. However, the analysis was conducted only at the constitutive level that, by its nature, is size-independent. The parameters obtained for these tests were applicable only to the particular specimen size analyzed.

A nonlocal microplane model (see ACI 446.1R) was used to analyze a rectangular concrete tension specimen in plane strain. The specimen was loaded by a prescribed uniform displacement and analyzed using two different meshes (Bazant and Ozbolt 1990, Droz and Bazant 1989). The model can represent tensile cracking in multiple directions, as well as nonlinear triaxial behavior in compression and shear, including post-peak response. The calculated load-displacement diagrams were obtained by local and nonlocal analysis for two meshes. While the local results differed substantially (spurious mesh sensitivity), the nonlocal results were nearly mesh independent. While producing excellent results, the nonlocal microplane model is very costly in terms of computer disk space and computer time.

Schlangen and van Mier (1992) employed a micromechanics approach to model tensile fracture using a triangular lattice of brittle beam elements with statistically varying properties to represent the heterogeneity of the material. Fracturing of the material takes place by removing, in each load step, the beam element with the highest stress (relative to its strength). The input parameters are, however, not directly observable material properties. Thousands of degrees of freedom are required for the analysis, and larger specimens would have required an unreasonable number of degrees of freedom.

Another lattice model, developed by Bazant et al. (1990) and refined by Jirasek and Bazant (1995a), involves lattices with only central force interactions between adjacent particles (avoiding the use of beams that are subject to bending and do not realistically reflect deformation modes on the microstructural level). The representation was further extended by Jirasek and Bazant (1995b) with a very efficient numerical algorithm of the central difference type. With this algorithm, it was possible to solve the nonlinear response of a lattice model with over 120,000 degrees of freedom by explicit dynamic integration on a desk-top workstation. The studies demonstrate that the transitional size effect can be reproduced by a lattice model.

Other examples of FEM analysis of tensile specimens are presented by Roelfstra et al. (1985), Rots (1988), Stankowski (1990), Stankowski et al. (1992), Vonk (1992), and Lopez and Carol (1995).

4.2.2 ***Compressive failure***

Although plain concrete compression specimens do not exhibit as significant a fracture mechanics size effect as notched tension specimens, a good model for concrete should be capable of predicting the behavior of such a specimen. A significant fracture mechanics size effect is found in the descending branch of the compression stress-strain curve (van Mier and Vonk 1991). Some fracture mechanics-based models have been used to attempt to predict the compressive strength of compression specimens. More test results exist for this type of specimen than for any other. All but one of the following examples used a smeared crack approach to analyze this specimen type.

Bazant and Prat (1988b) compared their local smeared crack microplane model for brittle-plastic materials with experimental uniaxial compression test results. Excellent agreement was found (although, once again, the five microplane model parameters were selected to provide the best fit to the experimental data). As with the tensile experiments, calculations with this model were purely constitutive, and therefore size independent and fracture insensitive. To obtain some size effect in the softening branches with the model as described (without going into a non-local implementation), a new set of parameters would be needed for each specimen size.

Gajer and Dux (1990) used a "simplified nonorthogonal crack model" to analyze a plain concrete prism in compression. The model was based on the crack-band concept, featuring simplified representations of both nonorthogonal cracking and the influence of cracking on local material stiffnesses. At an integration point, only one primary crack of fixed direction and one rotating crack are allowed to develop. The model makes use of the crack band concept to assure constant energy release for elements of different sizes during the fracture process. The model would be expected to give correct fracture mechanics results as long as the element size is larger than the width of the FPZ and smaller than the length of the FPZ.

Tasdemir, Maji, and Shah (1990) modeled mixed-mode crack initiation and propagation in the cement paste matrix, starting from the interface of a single rigid rectangular inclusion in a concrete specimen under uniaxial compression, using both a discrete LEFM approach and a FCM approach, with singularities modeled at the crack tips in both approaches. The FEM results were compared with experimental results and satisfactory local cracking results were obtained. The approach is applicable for the study of micromechanical behavior in the neighborhood of a single aggregate particle, but inappropriate as a method for determining the gross response of the compression specimen.

The original microplane model of Bazant and Prat (1988a, 1988b) was reformulated as a continuum damage model (Carol et al. 1991) and modified to make it computationally more efficient (Carol et al. 1992). In both cases, the original results of uniaxial compression specimens were reproduced with similar accuracy. The reformulation of the model in terms of a damage tensor, completely independent of the material rheology, allowed the authors to then replace elasticity with linear aging viscoelasticity and obtain a realistic approach to Rush curves (Carol and Bazant 1991, Carol et al. 1992). As discussed earlier, these results were purely constitutive and therefore size-independent and fracture-insensitive.

Bazant and Ozbolt (1992) used the nonlocal microplane model to investigate the effects of boundary conditions and size on the strength of a plane strain compression specimen. Meshes with 180 four-noded finite elements, enough to capture strain localizations, were employed. Many interesting and plausible damage localization patterns were displayed. Although their model is capable of predicting a fracture mechanics size effect, they found no size effect for this specimen type. The model would require great computational effort for realistic engineering structures.

Ozbolt and Bazant (1992) employed a nonlocal microplane model to simulate a uniaxial compression specimen subject to cyclic load. The rate effect was introduced by combining the damage model on each microplane with the Maxwell rheologic model. Nine four-noded isoparametric plane stress elements were employed. The model was shown to produce qualitatively correct results for cyclic compression, including hysteresis loops. This model, probably the most physically complete mechanical model developed for plain concrete to this time, requires a large computational effort for the analysis of even simple engineering structures.

Additional microstructural analyses of uniaxial compression specimens are reported by Stankowski (1990), Stankowski et al. (1992), and Vonk (1992, 1993).

4.2.3 *Fracture specimens*

The unreinforced three-point bend specimen is an important bench mark problem for several reasons. First, LEFM solutions for some span-to-depth ratios are known to better than 0.5 percent accuracy (Tada et al. 1973). Second, it is a standard fracture toughness testing geometry, which means that experimental results are available (Kaplan 1961, Catalano and Ingraffea 1982, Jenq and Shah 1985). Third, it is a symmetrical Mode I problem, and thus, the crack trajectory is known in advance. Symmetry can be used to reduce the cost of analysis. Many finite element analyses of this specimen are reported in the literature. It should be noted that most of the reported analyses were of relatively small specimens, and therefore LEFM would not be expected to apply.

The first analysis using the fictitious crack model of a three-point bend specimen was reported by Hillerborg, Modeer, and Petersson (1976). Finite element results regarding the effect of beam depth on flexural strength were presented. These theoretical results indicated that the flexural strength decreases with increased depth of the beam and that nonuniform shrinkage strains have a greater effect on the strength of deep beams than on the strength of shallow beams. A comparison between the theoretical predictions and a large number of experimental results showed good agreement. Similar analyses were presented by Modeer (1979), Petersson and Gustafsson (1980) for a discrete crack model and by Leibengood, Darwin, and Dodds (1986) for a smeared crack model.

Gerstle (1982) performed a series of discrete cracking analyses, employing a nonlinear FCM analysis, together with an LEFM fictitious crack tip assumption $K_{Ic(tip)}$. Nonlinear interface elements were used to model the fictitious crack. A secant stiffness iterative scheme was used to solve the nonlinear equations. Satisfactory agreement with experimental results (Catalano and Ingraffea 1982) was found using a pure FCM. When it was assumed that $K_{Ic(tip)}$ was nonzero, however, the strength of the beam was significantly over-predicted.

Gustafsson (1985) analyzed a three-point bend specimen using the FCM. He used a superposition of nonlinearly varying cohesive nodal forces to represent the fictitious crack, and a substructuring technique to reduce the number of unknowns in the iterative solution. He found the results to be consistent with theoretical expectations regarding the state of initial stress and specimen size.

De Borst (1986) analyzed a pre-notched, three-point bend specimen. The example was used mainly to demonstrate the efficiency of quasi and secant-Newton methods in the solution of nonlinear finite element equations. A local, smeared crack analysis was performed. He discussed several interesting points regarding the stability of the solution, and developed arc length control algorithms to increase solution stability for problems in which the post-peak response is important.

A three-point bending test according to a RILEM recommendation was numerically simulated using a discrete FCM with a bilinear stress versus crack opening displacement curve by Carpinteri et al. (1987) and Carpinteri (1989). The fictitious crack was represented by closing forces computed through the solution of a set of nonlinear equations. Many load-deflection curves for beams of various size were calculated and reported.

Yamaguchi and Chen (1990) used a smeared crack, crack-band-like model to simulate a notched three-point bend specimen. Eight-node isoparametric quadrilateral elements with 2 x 3 Gauss quadrature (3 in direction of crack propagation) were used in this analysis, and the notch was modeled as an element-wide gap. An extremely coarse mesh was used, with only two elements to model the 100 mm (3.9 in.) deep ligament. Good agreement was found with previous analysis results by others.

Gopalaratnam and Ye (1991) performed a fictitious crack analysis of a three-point bend specimen to determine the characteristics of the nonlinear FPZ in concrete. They used a linearized solution process to solve the nonlinear equations obtained in the FCM. The fictitious crack was modeled through appropriate application of cohesive point loads along the fictitious crack. Using a very refined mesh, they were able to obtain results that demonstrate the fracture mechanics size effect. They report extensively upon the process zone characteristics calculated.

Liang and Li (1991) used a boundary element program based on a discrete FCM to study the size effect in a plain concrete beam with a bending crack. The size effect was observed and verified. Justification for R-curve analysis (see ACI 446.1R) for such specimens was given. A double cantilever beam specimen is also analyzed using the same techniques.

Bolander and Hikosaka (1992) used a nonlocal smeared crack finite element model to simulate fracture of a three-point bend specimen. As shown by their finite element mesh, a large number of finite elements is required near the crack tip to adequately represent the development of the FPZ. They were able to obtain good agreement with experimental results using this approach, but it should be noted that, in their analysis, the FPZ was of significant size compared to the size of the beam specimen. Had the specimen been larger, the approach might not have been feasible because of the large number of elements required.

A nonlocal smeared crack analysis of a beam was reported by Bazant and Lin (1988). Geometrically similar notched beams of three different sizes of ratio 1:2:4 were analyzed using the meshes of four-node quadrilaterals. The concrete was described by the nonlocal smeared crack model, which is the same as the classical smeared crack model, except that the cracking strain was calculated from the maximum principal value of the spatially averaged nonlocal strain, $\bar{\varepsilon}$, rather than local strain, ε. The stress was assumed to decrease, either linearly or exponentially, with the maximum nonlocal principal strain. The characteristic length was taken in the first case as 2.3 in. (58 mm) and in the second case as 3.2 in. (81 mm), while the maximum aggregate size is d_a = 0.5 in. (13 mm). The maximum values of the nominal stress σ_N (load divided by beam depth and thickness) obtained by nonlocal finite element analysis with step-by-step loading are plotted as a function of the beam depth relative to the size of the aggregate, d/d_a, for both linear and exponential softening laws. Their work demonstrates that a nonlocal finite element model can represent the size effect quite well.

Bazant and Lin (1988) also compared results obtained with slanted square meshes to results obtained with aligned meshes. When the element size was less than about $^1/_3$ of the characteristic length L, they observed no mesh bias with regard to crack direction. In contrast, with the crack band model, it was impossible to analytically produce fractures in specimens with slanted meshes that run vertically, as they should.

Additional analyses of three-point bend specimens are reported by Rots (1988), Carol et al. (1993), Garcia-Alvarez et al. (1994), and Lotfi and Shing (1994).

4.2.4 *Shear failure in plain concrete beams*

The finite element analysis of the single-notched beam subject to four-point loading is a benchmark problem that has been investigated by many researchers. It provides a good test of the ability of a finite element code to model mixed-mode fracture. Tests have been performed by Arrea and Ingraffea (1982). No accurate closed-form LEFM solution to this problem has been reported. The following are descriptions of several of these analysis attempts.

Ingraffea and Gerstle (1985) performed several mixed-mode FCM analyses of the beam tested by Arrea and Ingraffea (1982) using interface elements. They used a discrete aggregate interlock model to simulate shear stress transfer across the fictitious crack. They also used quadratic elements, requiring a total of approximately 600 degrees of freedom. The predicted crack trajectory and load-displacement curve were in close agreement with the experimental results.

Ingraffea and Panthaki (1986) performed a FCM mixed-mode fracture analysis of the four point shear beam tested by Bazant and Pfeiffer (1985). They concluded that, although shear fracture (fracture forming by shear slip, rather than by tensile opening) can occur under certain conditions, tensile and not shear fracture occurred in the specimens.

A smeared crack analysis of the four point shear beam tested by Arrea and Ingraffea (1982) was performed by de Borst (1986). A rotating crack band model was used, with adjustment of the constitutive model with element size to maintain a constant fracture energy. Loading was controlled by the arc-length method to prevent solution instabilities. The analysis required approximately 600 degrees of freedom, using eight-noded elements. The finite element results agree closely with the experimental results.

A smeared crack band model that covers tensile softening in Mode I and shear softening in Mode II fracture was used by Rots and de Borst (1987) to analyze the specimen tested by Arrea and Ingraffea (1982). Interesting snap-back behavior was modeled using an arc-length control procedure.

Carpinteri (1989) used a FCM to simulate a double-edge notched four-point loaded shear specimen. Snapback instability was modeled for some cases, using a crack length control approach. A number of other problems were also analyzed.

A discrete cracking FCM was used by Gerstle and Xie (1992) to model the single-edge notched four-point loaded specimen of Arrea and Ingraffea (1982). A dynamic relaxation solution scheme was employed, with stress-based criteria for crack propagation and an automated remeshing algorithm. The analysis is significant in that it uses a very coarse mesh (200 degrees of freedom), yet seems to capture the significant behavior.

Malvar (1992) modeled the Arrea and Ingraffea (1982) beam using a local smeared crack approach with and without considering the transfer of shear stresses across the crack. A better prediction of the experimental results was obtained when shear stress transfer was modeled. However, up to the peak load, the results were not sensitive to presence or absence of shear stress transfer across the crack. Finally, it was shown that inadmissible results are obtained if both tensile and shear stresses are assumed to completely vanish upon cracking.

Schlangen and van Mier (1992) accurately modeled shear fracture with the triangular lattice model of brittle beam elements described in Section 4.2.1, which represent the heterogeneity of the material.

In conclusion, the four-point loaded shear beam has been analyzed by numerous investigators. The discrete cracking, the nonlocal smeared cracking, and the triangular lattice model all performed well in simulating the fracture behavior of the beam. All of the methods have particular advantages and limitations, and it is therefore difficult to decide which of the approaches is preferable although the discrete cracking approach appears to be more efficient for beams that approach the size at which LEFM becomes applicable.

4.2.5 *Dams*

Dams belong to an important class of concrete structures that are fracture sensitive, and may in some cases even be analyzed using LEFM. A collection of nine papers presented at the International Conference held in Vienna in 1988 concerning fracture of dams is given in a special issue of *Engrg. Frac. Mech.*, V. 35, 1/2/3, 1990. Other valuable sources include *Dam Fracture* (1991, 1994) and six papers on fracture of dams in *Fracture* (1987).

Several finite element studies of dams (considering cracking, but not using fracture mechanics concepts) were conducted in the 1960's, notably by Clough (1962) and Zienkiewicz and Cheung (1964).

Apparently, the first true FEM fracture mechanics analysis of a dam is presented by Ingraffea and Chappel (1981). The Fontana Dam had developed a crack, and finite element analyses using LEFM were used to model the discrete crack. A rational explanation for the crack was thus obtained.

A discrete crack FEM approach was employed by Skrikerud (1986) with a simple remeshing scheme to analyze a dam under dynamic excitation. However, despite his use of a discrete crack approach, no fracture mechanics based criteria were used to predict crack extension, and tensile stresses at the crack tip were simply compared with the tensile strength. This approach would produce nonobjective results, because the stress magnitudes produced by the crack tip elements would be strongly dependent upon their size.

A comprehensive investigation into the behavior of cracked concrete gravity dams using fracture mechanics concepts is presented in Saouma, Ayari and Boggs (1987). The work includes the results of discrete LEFM fracture mechanics analyses. The effect of various forms of loading, concrete age and anisotropy on the stress intensity factors, direction of crack profiles, crack lengths and stress redistribution is considered.

A plane strain LEFM discrete cracking finite element analysis of the Kolnbrein dam, a doubly curved arch dam in Austria, is presented by Wawrzynek and Ingraffea (1987) and Linsbauer et al. (1989a, 1989b). Cracks are loaded by water pressure, as well as by remotely applied stress resultants from a three-dimensional arch analysis. The cracks are approximately 14 m long, and therefore the assumption of LEFM seems justified. Actual fractures in the dam seem to have been accurately modeled using LEFM.

Cervera et al. (1990) describe a smeared crack approach to the analysis of progressive cracking in large dams. They account for crack pressurization by the water by including an effective stress in the model. Thermal and water pressure effects are also accounted for. 2D and 3D models are presented.

Ayari and Saouma (1990) developed discrete LEFM cracking models for the efficient simulation of transient dynamic discrete crack closure and crack propagation in dams. After presenting simple validation problems, these models are integrated into an interactive graphical program that was used to analyze the Koyna Dam.

Ingraffea (1990) describes the analyses of the Fontana Dam, a generic gravity dam, and the Kolnbrein Dam. The studies employ mixed-mode LEFM implemented within a discrete crack model, automatic rezoning, finite element method. The study of the Fontana dam elucidated the mechanisms for crack initiation, accurately reproduced the observed trajectory, and evaluated the effectiveness of interim repair measures. The study of the generic gravity dam has as its objective the evaluation of usefulness of LEFM for design and quality control during construction. An envelope of safe lengths, heights, and orientations of cracks potentially growing from cold lift joints on the upstream face is derived. It is also shown that, by neglecting the toughness of the foundation contact, classical design methods predict a conservative factor-of-safety against sliding, and that, when toughness is set to zero, LEFM predictions are in good agreement with the classical method.

Carpinteri et al. (1992) performed testing and analysis of a gravity dam. Two scaled-down (1:40) models of a gravity dam were subjected to equivalent hydraulic and self-weight loading. From an initial notch, a crack propagated during the loading process towards the foundation. The numerical simulation of the experiments used a FCM. The structural behavior of the models and the crack trajectories are reproduced satisfactorily.

4.2.6 *Bond-slip*

Modeling of bond-slip is one of the most difficult and controversial aspects of finite element analysis of reinforced concrete structures. Much of the controversy lies in the fact that many finite element models of reinforced concrete have provided excellent representations of experimental behavior while allowing no bond-slip to occur (i.e., perfect bond) (Darwin 1993). While it is known that bond-slip behavior arises at least partially from fracture of the concrete, other types of nonlinearity such as crushing in front of ribs, chemical adhesion, and friction between the concrete and the steel play a role. The detailed modeling of bond-slip associated with even one reinforcing bar is an extremely complex problem.

The problem of modeling the mechanical interaction between a ribbed steel reinforcing bar and the surrounding concrete has been addressed in a number of different ways. When the steel is modeled as smeared, the reinforced concrete is considered as a homogeneous composite material with constitutive relations that include (implicitly or explicitly) the bond-slip behavior. These models often represent perfect bond. At a more detailed level, individual reinforcing bars may be represented by truss elements. In this case, either the truss elements are connected directly to the concrete elements, where bond-slip is zero, or nonlinear link elements are used to connect the steel truss elements to the concrete to represent the effect of bond-slip. This is a cumbersome modeling approach, theoretically unsatisfactory from a fracture mechanics viewpoint, and nearly always ignores the splitting effects of bar movement. At a much more detailed scale, both the steel and the concrete can be modeled as continua, with interface elements placed between the concrete and the steel. Recognizing bond-slip as a fracture mechanics problem, the cracking of the concrete can be modeled using either discrete or smeared fracture mechanics. In this approach, the concrete can be considered as an unreinforced continuum surrounding the reinforcing bar. Several analyses of this type are described next.

In pioneering work, a singly reinforced, axisymmetric, “tension-pull” specimen was analyzed using the FEM by Bresler and Bertero (1968). To simulate cracking and other nonlinear effects, a soft “homogenized boundary layer” of prespecified width was assumed surrounding the steel. As expected, the existence of the boundary layer significantly decreased the overall stiffness of the specimen. However, this approach did not use fracture mechanics principles. The approach was of limited predictive utility, as both the width of the boundary layer and its constitutive properties were not determined from any fundamental theoretical or experimental information.

The first fracture mechanics approach to the analysis of bond slip was taken by Gerstle (1982) and Ingraffea et al. (1984). An axisymmetric discrete FCM analysis of a single concentrically reinforced axisymmetric tension-pull specimen was presented. No slip was assumed between the concrete and the steel; instead, “bond-slip” occurred as the result of the development of radial cracks propagating outward from each of the ribs on the reinforcing bar. The effect of longitudinal splitting cracks was neglected due to the axisymmetric modeling assumptions.

Vos (1983) reported on several smeared crack FEM bond-slip calculations. To account for the complex cracking/crushing situation surrounding a ribbed steel bar, the bar was assumed to be surrounded by a softening layer of concrete, approximately one bar diameter in width. The elements within this softening layer were assumed to be equal in length to the rib spacing on the bar. This approach suffers from the fact that the input parameters used to describe the softening layer are not based on fundamental or experimentally obtainable mechanical properties (such as tensile strength and fracture toughness). Additionally, mesh sensitivity would be expected because of the (local) softening model used to describe the concrete continuum. However, very close agreement between analysis and experiment was obtained.

Kaiser and Mehlhorn (1987) studied and compared the bond-link and the interface element approaches for modeling bond-slip between concrete and a single steel reinforcing bar. They analyzed a reinforced tensile specimen and compared the numerical solution with tests and the numerical solutions of others. They made the questionable assumption that bond-slip can be considered to be a property of the interface, rather than a property that results from damage to the structure.

Kay et al. (1992) developed a smeared crack model to represent accumulated damage in plain concrete and bond between concrete and steel reinforcing bars. The model used a continuum approach to describe microcracks and crack coalescence in plain concrete for the prediction of concrete tensile fracture. Concrete-reinforcing bar interaction was achieved through the use of one-dimensional beam elements that interact with three-dimensional continuum elements through a one-dimensional contact algorithm. Stick/slip interactions between the concrete and the reinforcing bars were expressed in terms of the interface stress or the internal damage variables of the concrete damage model.

Ozbolt and Eligehausen (1992) analyzed the pullout of a deformed steel bar embedded in a concrete cylinder under monotonic and cyclic loading. The analysis was performed using axisymmetric finite elements and a 3D microplane (smeared crack) model for concrete. Instead of the classical interface element approach, a more general approach with spatial discretization modeling the ribs of a deformed steel bar was employed. The pull-out failure mechanism was analyzed. Comparison between numerical results and test results indicated good agreement. Their approach was able to correctly predict the monotonic as well as cyclic behavior, including friction and degradation of pull-out resistance due to the previous damage.

Brown et al. (1993) and Darwin et al. (1994) modeled the bond-slip/strength behavior of individual bars in beam-end specimens. The 3D analyses represented the individual ribs on the bar, and used the FCM to represent concrete fracture and a Mohr-Coulomb model to represent steel-concrete interaction at the face of the ribs. The analyses, aimed at better understanding bond behavior, gave an excellent match with empirical relationships between bond strength and the effects of embedded length and concrete cover.

More work is required to develop credible methods for the numerical modeling of bond-slip.

4.2.7 ***Other types of plain concrete structures***

Several other types of plain concrete structures have been extensively analyzed using fracture mechanics and FEM approaches. Pullout of anchor bolts is an important fracture-controlled problem. Analyses of this problem has been reported by Hellier et al. (1987), Bittencourt, Ingraffea and Llorca (1992), Eligehausen and Ozbolt (1990), Cervenka et al. (1991), Eligehausen and Ozbolt (1992). Elfgren and Swartz (Elfgren 1992) have reported results of a round-robin analysis of anchor bolts organized by RILEM TC 90-FMA. Fifteen of the sixteen analyses were based on fracture mechanics concepts. Variations in the analysis assumptions included: LEFM, FCM, and smeared crack models. The responses included some 82 solutions. Punching shear in slabs and anchorage of tendons in prestressed members can also often be considered as the fracture of a plain concrete structure.

Other types of plain concrete structures that have been analyzed include thick walled rings (Pukl et al. 1992) and many analyses of the double cantilever beam (DCB) fracture toughness testing specimen (Gustafsson 1985, Sluys and de Borst 1992, Liang and Li 1991, Liaw et al. 1990a, Liaw et al. 1990b, Dahlblom and Ottosen 1990, Du et al. 1990, Bertholet and Robert 1990).

4.3—Reinforced concrete

The fracture mechanics analysis of reinforced concrete structures is often much more difficult than the analysis of plain concrete structures. The reinforcing steel may be modeled discretely (usually using truss elements) or in a smeared fashion, with the steel included in the constitutive model used to calculate the element stiffness matrices. In both cases, it is difficult to correctly model the effect of bond-slip between the concrete and the steel, although modeling bond-slip does not appear to be critical to the solution of many problems (Darwin 1993).

Often, but not always, the addition of steel reinforcement to a structure makes it fracture insensitive. Because of this fact, early finite element analyses of reinforced concrete structures were often successful, even without realistically modeling fracture of the concrete. Sometimes, however, structures will fail by fracturing in unanticipated ways. This is the reason for developing finite element methods that are capable of predicting fracturing modes of behavior. A summary of recent techniques used to model reinforced concrete structures using finite element analysis is presented by Darwin (1993).

In the remainder of this section, the literature is surveyed to illustrate various methods of analysis, including fracture mechanics effects, of reinforced concrete structures. In the course of the chapter, the term "tension stiffening" will be used to describe some models. The term refers to a numerical device used to limit the rate at which the stress across a smeared crack drops to zero once a crack forms. Tension stiffening represents the physical behavior of cracks crossed by reinforcing steel and serves to stabilize the numerical solution. Models that include tension stiffening usually do not take into account the fact that the method inherently represents fracture energy. Therefore, application of tension stiffening does not provide an accurate fracture mechanics representation and serves principally to slow the rate at which residual stresses are reimposed on the structure being modeled.

4.3.1 *Reinforced concrete membranes*

Shear walls in buildings are usually considered as membrane elements because the major loads are in the plane of the wall. Cervenka and Gerstle (1972) reported on an early smeared crack analysis of reinforced concrete panels that compared very favorably with their experimental results under monotonic loading but not under cyclic loading. Load-displacement diagrams, crack patterns and failure mechanisms of shear wall specimens were examined. The excellent agreement with the monotonic results probably resulted from the fact that these panels had a reasonable amount of reinforcing steel, and therefore were insensitive to fracture mechanics and did not exhibit the fracture mechanics size effect. The cracking of concrete in these panels was of a distributed, rather than a localized, nature. The lack of agreement under cyclic load was due to other (noncrack) aspects of their modeling scheme.

Another (local) smeared crack analysis of several Cervenka and Gerstle (1972) shear panels was conducted by Darwin and Pecknold (1976), using both smeared crack and smeared modeling of steel. The observed behavior of the panels was replicated very well by the analysis for both monotonic and cyclic loading. Other (local) smeared crack analyses of the same shear panel are presented by Schnobrich (1977) and Bergan and Holand (1979), giving excellent agreement between experimental and analytical results.

Bazant and Cedolin (1980) performed a smeared crack analysis of a center cracked reinforced concrete membrane in uniaxial tension. They did not, however, compare the analytical results to tests. They used the crack band model to investigate the effect of mesh size on the results. They observed that, if bond-slip is not correctly modeled, lack of objectivity results. In a subsequent paper, Bazant and Cedolin (1983) investigated the same problem, this time with respect to meshes at an angle to the crack propagation direction. They concluded that the blunt crack band approach is valid only if fine meshes are used, although correct convergence appears to occur as the mesh size tends to zero.

Gerstle (1982) and Ingraffea et al. (1984) employed a discrete FCM model to simulate a cracked membrane in uniaxial tension. Bond-slip was modeled using special "tension-softening" elements which bridge the primary crack. The effect of varying assumptions regarding bond-slip were studied.

Bedard and Kotsovos (1985) used a (local) smeared crack approach to model four reinforced concrete panels subjected to loading in pure shear, shear and uniaxial compression, and shear and biaxial compression. The panels were reinforced with different steel ratios in each direction, varying from 0.00713 to 0.01785. Good correlation was obtained with experimental results. In this case, fracture was not a controlling mechanism, and therefore correct modeling of fracture was unnecessary to obtain reasonable results.

Chang, Taniguchi and Chen (1987) applied a (local) smeared crack model to the analysis of reinforced concrete panels tested and analyzed by Vecchio (1986). The "fracture criteria" in the sophisticated constitutive model were based upon limiting stresses and strains, and therefore are not true fracture criteria. However, very good agreement between experiment and numerical results was obtained. The panels were heavily reinforced, and therefore fracture criteria, again, probably did not control the behavior of the panels.

Channakeshava and Iyengar (1988) presented a (local) smeared crack constitutive model used to model shear panels tested by Vecchio (1986). An elasto-plastic cracking model for reinforced concrete was presented. The model included concrete cracking in tension, plasticity in compression, aggregate interlock, tension softening, elasto-plastic behavior of the steel, bond-slip, and tension stiffening. A procedure for incorporating bond-slip in smeared steel elements was described. Good agreement with experimental results was obtained.

Barzegar (1989) examined the effect of skew, anisotropic reinforcement on the post-cracking response and ultimate capacity of reinforced concrete membrane elements under monotonically increasing proportional loading. Appropriate constitutive models, including post-cracking behavior, were discussed. Several shear panels were analyzed and compared with experimental results.

Crisfield and Wills (1989) used smeared crack models to analyze the concrete panels tested by Vecchio and Collins (1982). Different models involved both fixed and rotating cracks, with and without allowance for the tensile strength of concrete. A simple plasticity model was also applied. Because the numerical solutions were obtained using the arc-length analysis procedure, the complete (including post-maximum) load-deflection responses were traced, and consequently it was possible to clearly identify not only the computed collapse load but also the collapse modes.

Gupta and Maestrini (1989a, 1989b) used a (local) smeared approach to model the post-cracking behavior of reinforced concrete panels tested by Vecchio (1986). Good agreement was obtained.

Massicote, Elwi and MacGregor (1990) presented a practical two-dimensional hypoelastic model for reinforced concrete, with emphasis on a new approach for the description of tension stiffening, using a smeared crack band model. The model predictions were compared with test results of reinforced concrete members in uniaxial tension, reinforced concrete panels in shear, and reinforced concrete plates, simply supported along four sides and loaded axially and transversely.

Wu, Yoshikawa and Tanabe (1991) presented a complex smeared crack and damage mechanics approach used to model the tension stiffening effect for cracked reinforced concrete. Several membranes in various states of shear and tension were analyzed, and the results were compared with the experimental results. Nonlocal effects appear to have been ignored.

4.3.2 *Beams and frames*

A number of finite element analyses of beams and frames include correct fracture mechanics effects. In general, early models did not account for the fracture effects correctly, but since about 1980, most of the work has been quite sophisticated in this regard. Neither the smeared nor the discrete approach to cracking is clearly dominant. Some of the studies described below have provided useful insights into the structural behavior of beams. However, it is clear that the fracture analysis of beams and frames is not yet at a state where it can be used routinely in the design office.

Ngo and Scordelis (1967) reported the first discrete cracking FEM analysis of a reinforced concrete beam. Cracks were modeled but not propagated; their positions were assumed a priori. The analysis was linear elastic, and stresses at the crack tips were not accurately modeled. This pioneering paper did not make use of fracture mechanics principles.

Nilson (1967, 1968) discussed the use of the finite element method to represent reinforced concrete beams, including bond-slip and discrete cracking. He suggested an approach to modeling discrete cracks by remeshing to represent the development of cracks. His criteria for cracking were stress-based, rather than fracture mechanics-based, and yet, for the example that he studied (eccentrically reinforced tension-pull specimen), reasonable results were obtained. (The behavior of the tension-pull specimen that he studied was not primarily controlled by fracture of the concrete, but rather by the stiffness of the reinforcement.)

Valliapan and Doolan (1972) discussed a smeared crack approach to the finite element analysis of reinforced concrete. Their model assumed elastoplastic response of the concrete with a tension cut-off, discrete modeling of the steel using truss elements, and no bond-slip. They analyzed several reinforced concrete beams with reasonable success, even without using correct fracture mechanics principles.

Colville and Abbasi (1974) employed an approach very similar to that of Valliapan and Doolan (1972), with the exception that the steel was considered to be smeared. For the examples presented in the paper, reasonable results were obtained. Again, their examples appear to have been fracture insensitive.

Nam and Salmon (1974) reported upon the smeared crack analysis of reinforced concrete beams. Their approach was similar to that of Colville and Abassi (1974). One of their most significant conclusions was that an incremental tangent stiffness rather than an iterative initial stiffness method of solution was necessary to solve such problems. While cracks were predicted, fracture mechanics principles were not included in their model.

Salah El-Din and El-Adawy Nassef (1975) were among the first to combine fracture mechanics and the finite element method to analyze reinforced concrete beams. They assumed (incorrectly) that the conditions of LEFM apply to relatively small concrete beams. Discrete cracks were represented by setting the stiffnesses of the concrete elements equal to zero. They modeled the reinforcement discretely, and connected the steel elements directly to the concrete elements, modeling zero bond-slip. They used the compliance derivative method to calculate the energy release rate of a single vertical crack. Their derivation predicted the fracture mechanics size effect. By comparing their numerical results to the results of an experimental investigation, they found that the fracture mechanics-based approach was more accurate than the stress-based approach for the prediction of cracking in reinforced concrete beams.

Although they recognized that their smeared crack approach to the modeling of shear-critical reinforced concrete beams was not theoretically correct from a fracture mechanics viewpoint, Cedolin and Dei Poli (1977) concluded that their local nonlinear constitutive model for concrete was adequate to predict most of the significant responses. Studying a reinforced concrete beam 22 in. (560 mm) deep with approximately 1.8 percent flexural steel, they compared the numerical and experimental results. Their finite element model, which incorporated a nonlinear representation for concrete under biaxial stress, was able to predict the load-deflection curve, crack pattern, and failure load of the reinforced concrete beams. The model, however, was not able to represent dowel action and crack propagation at failure.

Bergan and Holand (1979), in a comprehensive discussion of finite element analysis of reinforced concrete structures, presented an analysis using local smeared crack techniques similar to those just described by Cedolin and Dei Poli (1977). The computed load-deflection curve agreed with the experimental results, except that the ultimate load was overpredicted, and the actual beam failed by a rapid diagonal tension failure mechanism that could not be captured by their smeared crack model.

A smeared crack approach was also employed by Bedard and Kotsovos (1985), who analyzed a reinforced concrete deep beam with web openings and web reinforcement, using plane stress analysis. Comparison with experimental results was good. They stated that the smallest finite element size should be greater than two to three times the maximum aggregate size. However, there is no evidence that such a rule would produce universally successful analyses.

Bazant, Pan, and Pijaudier-Cabot (1987) and Bazant, Pijaudier-Cabot, and Pan (1987) analyzed the softening post-peak load-deflection relation for reinforced concrete beams and frames using layered finite elements. Concrete was modeled as a strain softening material in both tension and in compression; the steel reinforcement was modeled as elastic-plastic. Standard bending theory assumptions were used and bond-slip of reinforcement was neglected. The model could approximate existing test results for beams and frames. At the same time, constitutive laws with strain softening, including those of continuum damage mechanics, were shown, in general, to lead to spurious sensitivity of results with respect to the chosen finite element size, similar to that documented for other strain-softening problems. In analogy to the finite element crack-band model (Section 3.2.1), they recommend the use of a minimum admissible element size, specified as a cross sectional property.

One of the major problems in the analysis of reinforced concrete structures is the representation of reinforcing bars. Typically, either the steel is represented in a smeared manner, in which case the concrete and steel are combined and treated as a composite material, or the steel is represented discretely, in which case nodes must be collocated with the steel bars, which severely constrains the finite element mesh. Allwood and Bajarwan (1989) presented an interesting approach for modeling reinforcing steel that allows the steel to be represented independently of the concrete (using separate nodes and elements). An iterative solution scheme is used to bring the steel elements into equilibrium with the concrete elements.

Balakrishnan and Murray (1988) described a practical stress-strain relationship for concrete for use in a smeared crack model, which was effective in predicting the behavior of reinforced concrete beams and panels. The model includes homogenized material properties that incorporate the estimated effects of strain localization. Methods of estimating these homogenized properties are presented in their paper.

Channakeshava and Iyengar (1988) used a smeared crack model for a singly reinforced concrete beam without shear reinforcement. The shear critical beam failed by diagonal cracking. In the analysis, although the exact mode of failure could not be assessed, extensive diagonal cracking with large crack strains was observed prior to failure. Their model took into account the width of the element/Gauss point over which a crack is assumed to be smeared, and thus attempted to use a form of the crack band model to objectively represent the fracture energy associated with discrete cracking. They concluded that an accurate response was obtained using the technique.

Gustafsson and Hillerborg (1988) used the finite element method to model size effects due to discrete cohesive diagonal tension cracks in singly reinforced beams without shear reinforcement. They found a significant fracture mechanics size effect. They carried out studies of the sensitivity of shear strength to reinforcement ratio and depth-to-span ratio. Their work, aside from being correct from a fracture mechanics viewpoint, serves as an example of how the finite element method can be used to great advantage in developing design code equations. More recently, Ma, Niwa, and McCabe (1992) performed similar fictitious crack analyses. Their results show conclusively that a size effect exists for shear critical beams. They therefore recommend that the ACI-318 code be modified to reflect this size effect.

Cervenka, Eligehausen, and Pukl (1991) used a smeared crack approach to model shear failure in simply supported, reinforced and unreinforced concrete beams without shear reinforcement, using a crack band model. They performed two plane stress analyses of a fracture-sensitive unreinforced beam. One analysis used a fine mesh, and the other used a coarse mesh. They found that the same peak load was predicted by both meshes, but that the post-peak behavior of the beam was sensitive to mesh density.

Feenstra, de Borst, and Rots (1991b) used the fictitious crack approach, including a representation for aggregate interlock, to model a moderately deep beam with several predefined discrete crack locations. Eight-noded continuum elements, six-noded interface elements and three-noded truss elements were used to model steel, with interface elements used to model bond-slip. A Newton-Raphson method with indirect displacement control was used. The computed response of the beam matched the experimental results reasonably well. This, of course, was not a completely predictive analysis because they inserted fictitious cracks at predetermined locations.

Gajer and Dux (1990, 1991) analyzed two reinforced concrete beams using a crack band model, featuring simplified representations of both nonorthogonal cracking and the influence of cracking on local material stiffnesses. At an integration point, one primary crack of fixed direction and one rotating crack are allowed to develop. The beams analyzed by de Borst and Nauta (1985) and Bedard and Kotsovos (1985) were analyzed, with satisfactory results.

Ozbolt and Eligehausen (1991) studied the shear resistance of four similar reinforced concrete beams of different sizes without shear reinforcement using a nonlocal microplane model and plane stress finite elements to study the size effect. The calculated results demonstrated a decrease in average strength with an increase in member size. It was demonstrated that the results were insensitive to mesh size and load path, in contrast to the results of local continuum analyses. They stated that element size must not be larger than one-half of the characteristic length of the nonlocal continuum. Otherwise, the analysis is equivalent to the classical local continuum analysis. Thus, large numbers of elements were required in these analyses, especially for the larger beams.

Pagnoni, Slater, Ameur-Moussa, and Buyukozturk (1992) formulated a finite element procedure for the nonlinear analysis of general three-dimensional reinforced concrete structures. They employed a three-dimensional version of the crack band model to objectively represent fracture. The model was demonstrated using detailed analyses of a singly reinforced deep beam and a prestressed concrete reactor vessel.

4.3.3 *Containment vessels*

Concrete containment vessels present challenging analysis problems. However, because of the severe consequences of a failure, detailed analysis is required to assure a safe design. These vessels are usually heavily reinforced, and therefore, fracture mechanics issues are of secondary importance.

Rashid (1968), in the first application of the smeared crack approach, analyzed a reinforced concrete pressure vessel. The steel elements were assumed to be elastic-perfectly-plastic. Concrete and steel were modeled using separate finite elements and an axisymmetric condition was assumed. Limited experimental comparisons were made. Because the concrete cracking was expected to be distributed, rather than localized, due to the presence of heavy reinforcement, no fracture mechanics issues arose from this approach. In essence, the composite reinforced concrete material was fracture insensitive.

Meyer and Bathe (1982) used smeared crack and smeared steel models to address some of the questions facing the engineer charged with assessing the safety of concrete structures for which conventional linear analysis is deemed inadequate. Guidelines were given to aid the engineer in selecting the material model most appropriate for this purpose. Questions related to finite element analysis were discussed, including numerical solution techniques, their accuracy, efficiency and applicability. An example of a prestressed concrete reactor vessel illustrated the application of nonlinear finite element analysis to concrete structures in engineering practice.

Clauss (1987) presented a summary of the analyses of a one-sixth scale model of a reinforced concrete containment vessel. The analyses were conducted by ten organizations in the United States and Europe. Each organization was supplied with a standard information package, which included construction drawings and actual material properties for most of the materials used in the model. Each organization worked independently using its own analytical methods. The report includes descriptions of the various analytical approaches and pretest predictions. Significant milestones that occur with increasing pressure, such as cracking and crushing of the concrete, yielding of the steel components, and the failure pressure and failure mechanism were described.

4.3.4 *Plates and shells*

Plates and shells are common components of reinforced concrete structures. Cracking can significantly influence both the stiffness and strength of these members. To model such a structure as a three-dimensional solid is computationally expensive. It is therefore necessary to use special plate and shell finite elements. Various approaches have been taken to model the effects of reinforcement and cracking, as the following references demonstrate. In general, a close match has been obtained between analytical and experimental results, without special regard for fracture concepts. The lack of importance of fracture mechanics is likely due to the dominant role played by reinforcement in the structures analyzed.

Hand, Pecknold, and Schnobrich (1973) used layered elements, in which material properties changed from layer to layer, to model several reinforced concrete plates under uniform stress, a corner-propped reinforced concrete plate under a concentrated central load, and a reinforced concrete shell under uniform load. The numerical results closely match the experimental results for all problems. Cracking was governed by the principal tensile stress.

Lin and Scordelis (1975) took a layered shell element approach to model both the reinforcement and the concrete. The concrete was assumed to behave plastically in compression, and as a gradually softening material in tension. Three examples, including a circular slab, a square slab, and a hyperbolic paraboloid shell, were analyzed, with the predicted results closely matching the experimental data.

Schnobrich (1977) used a (local) smeared crack and smeared steel model to solve several plate and shell problems. The agreement between analysis and experiment was excellent. Specimens modeled include a plate subject to twist and uniaxial moment, a corner-supported two-way slab, a three-hole conduit, a shear panel, and a hyperbolic paraboloid shell roof subjected to vertical loading.

Bashur and Darwin (1978) used a (local) smeared crack model to represent cracking as a continuous process through the depth of plate elements. Reinforcing steel was represented as a smeared material at the appropriate position(s) through the depth. Element stiffness was determined using numerical integration, removing the need to use layered elements. A close match was obtained with experimental results for a series of one-way and two-way reinforced concrete slabs.

Bergan and Holand (1979) analyzed reinforced concrete plates, using a smeared crack and smeared reinforcing model. Initially, zero tensile strength was assumed, which produced deflections that were too large. A gradually softening tensile behavior, however, produced a good comparison with experimental data. Neither of the plates analyzed was fracture sensitive.

Barzegar (1988) introduced the use of a layering technique in general nonlinear FEM analysis of reinforced concrete flat shell elements. A stack of simple four-node plane stress and Mindlin plate elements were used to simulate membrane and bending behavior. The method, together with nonlinear constitutive models for concrete and steel, was described and implemented. Bond degradation between concrete and steel was simulated. Several examples were presented.

Hu and Schnobrich (1991) proposed plane stress constitutive models for the nonlinear FEM analysis of reinforced concrete structures under monotonic loading. An elastic strain hardening plastic stress-strain relationship with a non-associated flow rule was used to model concrete in the compression dominated region and an elastic brittle fracture behavior was assumed for concrete in the tension dominated area. After cracking took place, the smeared crack approach, together with the rotating crack concept, was employed. Together with layered elements, these material models were used to model the flexural behavior of reinforced concrete plates and shells. Good agreement with test results was obtained. Fracture mechanics issues were not considered.

Lewinski and Wojewodzki (1991) introduced a nonlinear (local, smeared crack) procedure to solve the elasticity problem for a doubly reinforced concrete slab was found. Ten different cracking patterns were assumed in this formulation. Use of each pattern implied the division of the slab thickness into several layers (maximum three layers of concrete). Numerical integration was used through the total thickness. Results obtained with this approach compared favorably with experimental results for deep beams and plates.

4.3.5 *Other reinforced concrete structures*

Gerstle (1987) used a plane strain (local) smeared crack finite element analysis technique for the analysis of reinforced concrete culverts. Since the steel was smeared, the reinforced concrete was treated as a composite material. No post-cracking strength was included in the model. The FEM analyses matched the experimental results reasonably well. However, the analyses tended to underestimate the deflections and overestimate the strength of the ring sections.

Swenson and Ingraffea (1991) used discrete cracking models to analyze the Schoharie Creek Bridge. The bridge collapsed in 1987 due to a flood that scoured the soil under a pier, which led to unstable cracking of the minimally reinforced pier section and precipitated the actual collapse. The analyses included linear and nonlinear fracture mechanics evaluations of the initiation, stability, and trajectory of the crack that precipitated failure. A principal motivation of the effort was to apply the results of recent concrete fracture mechanics research to a problem of practical significance. The authors concluded that cracking of the bridge foundation could be rationally explained through the use of fracture mechanics principles.

4.4—Closure

Over the past 30 years, research into finite element modeling of plain and reinforced concrete structures has been intense. The early models that did not take fracture mechanics into account proved quite satisfactory to solve the problems to which they were applied. While the possibility exists that the problems that have been presented in the literature were only the most successful (and fracture sensitive structures for which correspondence between analysis and test results was poor were simply not reported), it is clear that the published problems that were solved without the proper application of fracture mechanics were largely fracture insensitive. Further, nearly all of the analyses fail to include a convergence study to determine the effect of mesh refinement upon the solution.

Discrete and smeared crack FEM analyses have successfully solved problems that are known to be fracture sensitive. However, most of the problems solved to date have been almost trivial in their geometric simplicity. The challenge now is to apply correct fracture mechanics principles to the efficient analysis of nontrivial structures, such as reinforced concrete solids, frames, plates, and shells.

It does not appear that the discrete approach is “better” than the smeared approach or vice versa. On the contrary, it appears that it would be wise to move toward a marriage of these two approaches to better model the entire regime of concrete behavior from multiaxial compression of small specimens to LEFM in large structures. This is surely one of the major challenges of the coming years.

CHAPTER 5—CONCLUSIONS

5.1—General summary

The purpose of this report has been to describe the state-of-the-art of finite element analysis of fracture in reinforced concrete structures. Chapter 1 provides general background and historical perspective, and highlights the two most popular methods used in finite element analysis to represent fracture: the discrete crack approach and the smeared crack approach. Chapters 2 and 3 present the two approaches in greater detail. In each case, the goal has been to provide a broad view of the field and to provide references to the literature so that the interested reader may go directly to the primary sources for additional information.

Chapter 4 provides examples (with special attention to fracture mechanics) of finite element analyses of concrete structures that are available in the archival literature. Attention is focused on twelve types of structures, consisting of both plain and reinforced concrete. The review includes almost one hundred papers that describe finite element analyses of these structural types. While most of the early analyses of concrete were inadequate from a theoretical fracture mechanics viewpoint, the results proved more than adequate, probably because of the nature of the problems selected. These early models were principally research tools, not applicable to design. The more recent efforts, although largely theoretically correct from a fracture mechanics viewpoint, generally describe analysis methods that remain too cumbersome to apply in routine design environments. For the most part, only the most readily available archival literature has been referenced.

From Chapter 4, it is clear that concrete structures can be categorized into two classes: those with behavior that is not governed by fracture mechanics and those with behavior that is. It is clear that special fracture mechanics considerations are not necessary for a large class of reinforced concrete structures. However, it is also clear that some reinforced concrete structures are fracture sensitive. To predict fracture behavior, a finite element code must include the capability to model fracture. As far as the engineer is concerned, if the structure being analyzed is known to be fracture insensitive, then the analysis may proceed without resorting to fracture mechanics. On the other hand, for structures that are known to be fracture sensitive, the engineer is well-advised to use finite element codes that correctly model fracture. There are, of course, times when an engineer is uncertain whether or not a structure is fracture sensitive. In these cases, the use of a finite element code capable of predicting fracture is advisable.

Fracture sensitivity is similar to buckling in that both behaviors may or may not exist, but each can only be detected through the use of analytical tools capable of predicting such effects. It seems likely that, in the next ten years, reliable codes with the capability to predict fracture of concrete structures will be developed, just as reliable codes with geometrically nonlinear capabilities have developed over the past ten years.

5.2—Future work

Very few successful commercial finite element codes can handle the nonlinear analysis of concrete structures. Those commercial codes that can handle the analysis and design of reinforced concrete structures typically perform linear analyses, or at most nonlinear geometric analyses, and then use the results of these analyses, together with design provisions (ACI 318), to determine the adequacy of the members. In other words, current design practice does not account for nonlinear material behavior, much less fracture behavior, during the analysis phase.

It therefore seems that the research community is far ahead of (and out of step with) the practicing community in the analysis of concrete structures. Since practicing engineers do not have a routine need to predict fracture behavior, they are not willing to pay for such an analysis capability. When they do have a need to predict or evaluate the fracture behavior of a structure, it usually involves a special application requiring so much of their time and energy that they often go to researchers to provide the solution.

The research community should continue to develop generally accepted methods for numerically modeling fracture behavior. But it is clear that, at least for now, these analysis methods will not be used directly by practicing design engineers, but rather by other researchers who may use the codes to study aspects of concrete structural behavior. These codes may be used, for instance, to develop new and better design expressions for the ACI Building Code (ACI 318).

A fruitful area of research involves combining the smeared crack and the discrete crack approaches. Both approaches have limitations, and by combining the two approaches, a high-fidelity numerical representation of concrete behavior may be achieved.

The fracture analysis of three-dimensional structures is still in its infancy. Both the smeared and discrete crack approaches require further development to be useful for the analysis of three-dimensional solids, not to mention shells, plates, and beams. The numerical approaches to the representation of the interaction between reinforcement and concrete (bond-slip) are currently unsatisfactory and primitive. Also, the effects of concrete creep, shrinkage, and cyclic behavior are usually ignored in numerical models of concrete behavior, even though they may have large effects upon structural response. Although much progress has been made, the goal of developing accurate and general numerical models for plain and reinforced concrete structures remains.

CHAPTER 6—REFERENCES

6.1—Recommended references

American Concrete Institute

318 Building Code Requirements for Reinforced Concrete

446.1R Fracture Mechanics of Concrete: Concepts, Models and Determination of Material Properties

6.2—Cited references

Allwood, R. J., and Bajarwan, A. A., 1989, “A New Method for Modeling Reinforcement and Bond in Finite Element Analyses in Reinforced Concrete,” *Int. J. Num. Meth. in Engrg.*, V. 28, pp. 833-844.

Arrea, M., and Ingraffea, A. R., 1982, “Mixed-Mode Crack Propagation in Mortar and Concrete,” *Report* No. 81-13, Department of Structural Engineering, Cornell University, Ithaca, N.Y.

ASCE Task Committee on Finite Element Analysis of Reinforced Concrete Structures, 1982, *Finite Element Analysis of Reinforced Concrete*, A. H. Nilson, Chmn., American Society of Civil Engineers, New York, 545 pp.

Atluri, S. N.; Kobayashi, A. S.; and Nakagaki, M., Apr., 1975, “An Assumed Displacement Hybrid Element Model for Linear Fracture Mechanics,” *Int. J. Fracture*, V. 11, No. 2, pp. 257-271.

Ayari, M. L., and Saouma, V. E., 1990, “A Fracture Mechanics Based Seismic Analysis of Concrete Gravity Dams Using Discrete Cracks,” *Engrg. Frac. Mech.*, V. 35, No. 1/2/3, pp. 587-598.

Balakrishnan, S., and Murray, D. W., 1988, “Concrete Constitutive Model for NLFE Analysis of Structures,” *J. Struct. Engrg.*, ASCE, V. 114, No. 7, July, pp. 1449-1466.

Balakrishnan S.; Elwi, A. E.; and Murray, D. W., 1988, “Effect of Modeling on NLFE Analysis of Concrete Structures,” *J. Struct. Engrg.*, ASCE, V. 114, No. 7, July, pp. 1467-1487.

Barsom, J. M., and Rolfe, S. T., 1987, *Fracture and Fatigue Control in Structures*, 2nd Edition, Prentice-Hall, New York.

Barsoum, R. S., 1976, “On the Use of Isoparametric Finite Elements in Linear Fracture Mechanics,” *Int. J. Num. Meth. in Engrg.*, V. 10, pp. 25-37.

Barzegar, F., 1988, “Layering of RC Membrane and Plate Elements in Nonlinear Analysis,” *J. Struct. Engrg.*, ASCE, V. 114, No. 11, Nov., pp. 2474-2492.

Barzegar F., 1989, “Analysis of RC Membrane Elements with Anisotropic Reinforcement,” *J. Struct. Engrg.*, ASCE, V. 115, No. 3, Mar., pp. 647-665.

Bashur, F. K. and Darwin, D., 1978, “Nonlinear Model for Reinforced Concrete Slabs,” *J. Struct. Div.*, ASCE, V. 104, No. ST1, Jan., pp. 157-170.

Bathe, K.-J., 1982, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, N.J.

Bazant, Z. P., 1976, “Instability, Ductility and Size Effect in Strain-Softening Concrete,” *J. Engrg. Mech. Div*., ASCE, V. 102, No. EM2, Apr., pp. 331-344.

Bazant, Z. P., 1984, “Size Effect in Blunt Fracture: Concrete, Rock, Metal,” *J. Engrg. Mech.*, ASCE, V. 110, No. 4, Apr., pp. 518-535.

Bazant, Z. P., 1986, “Mechanics of Distributed Cracking,” *ASME Appl. Mech. Rev.*, V. 39, No. 5, May, pp. 675-705.

Bazant, Z. P., 1994, “Nonlocal Damage Theory Based on Micromechanics of Crack Interactions,” *J. Engrg. Mech.*, ASCE, V. 120, No. 3, Mar., pp. 593-617. Addendum and Errata, V. 120, No. 6, June, pp. 1401-1402.

Bazant, Z. P., and Cedolin, L., 1979, “Blunt Crack Band Propagation in Finite Element Analysis,” *J. Engrg. Mech.*, ASCE, V. 105, No. 2, Feb., pp. 297-315.

Bazant, Z. P., and Cedolin, L., 1980, “Fracture Mechanics of Reinforced Concrete,” *J. Engrg. Mech.*, ASCE, V. 106, No. 6, Dec., pp. 1287-1306.

Bazant, Z. P., and Cedolin, L., 1983, “Finite Element Modeling of Crack Band Propagation,” *J. Struct. Engrg.*, ASCE, V. 109, No. 2, Feb., pp. 69-92.

Bazant, Z. P., and Oh, B. H., 1983, “Crack Band Theory for Fracture of Concrete,” *Mater. Struct.*, RILEM, Paris, France, V. 16, Feb., pp. 155-177.

Bazant, Z. P.; Belytschko T. B.; and Chang, T.-P., 1984, “Continuum Theory for Strain-Softening,” *J. Engrg. Mech.*, ASCE, V. 110, No. 12, pp. 1666-1692.

Bazant, Z. P., and Pfeiffer, P. A., 1985, “Tests of Shear Fracture and Strain Softening in Concrete,” *Proc.*, 2nd Symposium on the Interaction of Non-Nuclear Munitions on Structures, Panama City Beach, Florida, Apr., 15-19.

Bazant, Z. P.; Pan, J.; and Pijaudier-Cabot, G., 1987, “Softening in Reinforced Concrete Beams and Frames,” *J. Struct. Engrg.*, ASCE, V. 113, No. 12, Dec., pp. 2333-2347.

Bazant, Z. P.; Pijaudier-Cabot, G.; and Pan, J., 1987, “Ductility, Snapback, Size Effect, and Redistribution in Softening Beams or Frames,” *J. Struct. Engrg.*, ASCE, V. 113, No. 12, Dec., pp. 2348-2364.

Bazant, Z. P., and Lin, F. B., 1988, “Nonlocal Smeared Crack Model for Concrete Fracture,” *J. Engrg. Mech.*, ASCE, V. 114, No. 11, Nov., pp. 2493-2510.

Bazant, Z. P., and Pijaudier-Cabot, G., 1988, “Nonlocal Continuum Damage, Localization Instability and Convergence," *J. Appl. Mech.*, ASME, V. 55, No. 2, pp. 287-293.

Bazant, Z. P., and Prat, P. C., 1988a, "Microplane Model for Brittle-Plastic Material: I. Theory," *J. Engrg. Mech.*, ASCE, V. 114, No. 10, Oct., pp. 1672-1688.

Bazant, Z. P., and Prat, P. C., 1988b, "Microplane Model for Brittle-Plastic Material: II. Verification," *J. Engrg. Mech.*, V. 114, No. 10, Oct., pp. 1689-1702.

Bazant Z. P., and Pijaudier-Cabot, G., 1989, "Measurement of Characteristic Length of Nonlocal Continuum," *J. Engrg. Mech.*, ASCE, V. 115, No. 4, Apr., pp. 755-767.

Bazant, Z. P., and Ozbolt, J., 1990, "Nonlocal Microplane Model for Fracture, Damage, and Size Effect in Structures," *J. Engrg. Mech.*, ASCE, V. 116, No. 11, Nov., pp. 2485-2505.

Bazant, Z. P.; Tabbara, M. R.; Kazemi, M. T.; and Pijaudier-Cabot, G., 1990, "Random Particle Model for Fracture of Aggregate or Fiber Composites," *J. Engrg. Mech.*, ASCE, V. 116, No. 8, Aug., pp. 1686-1705.

Bazant, Z. P., and Ozbolt, J., 1992, "Compression Failure of Quasibrittle Material: Nonlocal Microplane Model," *J. Engrg. Mech.*, ASCE, V. 118, No. 3, Mar., pp. 540-556.

Bazant, Z. P., and Jirasek, M., 1994, "Damage Nonlocality due to Microcrack Interactions: Statistical Determination of Crack Influence Function," *Fracture and Damage in Quasibrittle Structures: Experiment, Modeling, and Computer Analysis*, Z. P. Bazant, Z. Bittnar, M. Jirasek, and J. Mazars, eds., E&FN Spon, London, pp. 3-17.

Bazant, Z. P., and Li, Y.-N., 1995, "Stability of Cohesive Crack Model," *J. Appl. Mech.*, ASME, V. 62, No. 4, pp.

Bedard, C., and Kotsovos, M. D., 1985, "Application of NLFEA to Concrete Structures," *J. Struct. Mech.*, ASCE, V. 111, No. 12, Dec., pp. 2691-2707.

Belytschko, T.; Fish, J.; and Engelmann, B. E., 1988, "A Finite Element with Embedded Localization Zones," *J. Comp. Meth. Appl. Mech. Engrg.*, V. 70, pp. 59-89.

Bergan, P., and Holand, I., 1979, "Nonlinear Finite Element Analysis of Concrete Structures," *Comp. Meth Appl. Mech Engrg.*, V. 17/18, pp. 443-467.

Bertholet, J. M., and Robert, J. L., 1990, "Damage Evaluation of Concrete Test Specimens Related to Failure Analysis," *J. Engrg. Mech.*, ASCE, V. 116, No. 3, Mar., pp. 587-605.

Bittencourt, T. N.; Barry, A.; and Ingraffea, A. R., 1992, "Comparison of Mixed-Mode Stress Intensity Factors Obtained Through Displacement Correlation, J-Integral Formulation, and Modified Crack-Closure Integral," *Fracture Mechanics: Twenty Second Symposium*, ASTM STP 1131, American Society for Testing and Materials, Philadelphia, V. II, pp. 69-82.

Bittencourt, T.; Ingraffea, A. R.; and Llorca, J., 1992, "Simulation of Arbitrary, Cohesive Crack Propagation," *Fracture Mechanics of Concrete Structures*, edited by Z. Bazant, Elsevier, London, pp. 339-350.

Bocca, P.; Carpinteri, A.; and Valente, S., 1991, "Mixed Mode Fracture of Concrete," *Int. J. Solids Structures*, V. 27, No. 9, pp. 1139-1153.

Bolander, J., and Hikosaka, H., 1992, "Applying Nonlocal Smeared Crack Concepts to Modeling Concrete Fracture," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 198-203.

Bresler, B., and Bertero, V., 1968, "Behavior of Reinforced Concrete under Repeated Load," *J. Struct. Div*., ASCE V. 94, No. ST6, June, pp. 1567-1589.

Broek, D., 1986, *Elementary Engrg. Frac. Mech.*, 4th Ed., Klewer Academic Publishers, Dordrecht.

Brown, C. J.; Darwin, D.; and McCabe, S. L., 1993, "Finite Element Fracture Analysis of Steel-Concrete Bond," *SM Report* No. 36, University of Kansas Center for Research, Lawrence, Kansas.

Byskov, E., 1970, "The Calculation of Stress Intensity Factors Using the Finite Element Method with Cracked Elements," *Int. J. Fracture Mechanics*, V. 6, No. 2, June, pp. 129-267.

Carol, I., and Bazant, Z., 1991, "Damage-Rheology Uncoupling for Microplane Damage Tensor, with Application to Concrete with Creep," *Constitutive Laws for Engineering Materials*, C. Desai, E. Krempl, G. Frantziskonis, and H. Saadatmanesh, eds., ASME Press, Tucson, pp. 391-394.

Carol, I.; Bazant, Z. P.; and Prat, P. C., 1991, "Geometric Damage Tensor Based on Microplane Model," *J. Engrg. Mech.*, ASCE, V. 117, No. 10, Oct., pp. 2429-2448.

Carol, I.; Bazant, Z.; and Prat, P., 1992, "Microplane-Type Constitutive Models for Distributed Damage and Localized Cracking in Concrete Structures," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 299-304.

Carol, I.; Prat, P.; and Bazant, Z., 1992, "New Explicit Microplane Model for Concrete: Theoretical Aspects and Numerical Implementation," *Int. J. Solids Structures*, V. 29, No. 9, pp. 1173-1191.

Carol, I.; Prat, P.; and Gettu, R., 1993, "Numerical Analysis of Mixed-Mode Fracture of Quasibrittle Materials Using a Multicrack Constitutive Model," *Mixed-Mode Fatigue and Fracture*, edited by H. Rossmanith and K. Miller, ESIS Publ. 14, Mechanical Engineering Publications Ltd., London, pp. 319-332.

Carol, I., and Willam, K., 1994, "Microcrack Opening/Closure Effects in Elastic-Degrading Models," *Fracture and Damage in Quasibrittle Structures: Experiment, Modeling, and Computer Analysis*, Z. P. Bazant, Z. Bittnar, M. Jirasek and J. Mazars, eds., E&FN Spon, London, pp. 41-52.

Carpinteri, A.; Colombo, G.; Ferrara, G.; and Giuseppetti, 1987, "Numerical Simulation of Concrete Fracture Through a Bilinear Softening Stress-Crack Opening Displacement Law," *Fracture of Concrete and Rock*, Proc. SEM-RILEM Int. Conf., pp. 178-191.

Carpinteri, A., 1989, "Size-Scale Effects on the Brittleness of Concrete Structures: Dimensional Analysis and Snap-Back Instability," *Fracture Mechanics: Application to Concrete*, edited by V. C. Li and Z. P. Bazant, ACI SP-118, American Concrete Institute, Detroit, pp. 197-235.

Carpinteri, A.; Valente, S.; Ferrara, G.; and Imperato, L., 1992, "Experimental and Numerical Fracture Modelling of a Gravity Dam," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 351-360.

Catalano, D., and Ingraffea, A. R., 1982, "Concrete Fracture: A Linear Elastic Fracture Mechanics Approach," MS Thesis, Department of Structural Engineering Report 82-1, School of Civil and Environmental Engineering, Cornell University, Ithaca, NY.

Cavendish, J. C.; Field, D. A.; and Frey, W. H., 1985, "An Approach to Automatic Three-dimensional Finite Element Mesh Generation," *Int. J. Num. Meth. in Engrg.*, V. 21, pp. 329-347.

Cedolin, L., and Dei Poli, S., 1977, "Finite Element Studies of Shear-Critical R/C Beams," *J. Engrg. Mech. Div.*, ASCE, V. 103, No. EM3, June, pp. 395-410.

Cervenka, V., and Gerstle, K. H., 1971, "Inelastic Analysis of Reinforced Concrete Panels, Part I: Theory," *IABSE Publications*, V. 31-II, pp. 31-45.

Cervenka, V., and Gerstle, K. H., 1972, "Inelastic Analysis of Reinforced Concrete Panels, Part II: Experimental Verification and Application," *IABSE Publications*, V. 32-II, pp. 25-39.

Cervenka, V.; Eligehausen, R.; and Pukl, R., 1991, "Computer Models of Concrete Structures," *IABSE Colloquium*, Stuttgart, pp. 311-320.

Cervera, M.; Oliver, J.; Herrero, E.; and Onate, E., 1990, "A Computational Model For Progressive Cracking in Large Dams Due to the Swelling of Concrete," *Engrg. Frac. Mech.*, V. 35, No. 1/2/3, pp. 573-585.

Chan, S. K.; Tuba, I. S.; and Wilson, W. K., 1970, "On the Finite Element Method in Linear Fracture Mechanics," *Engrg. Frac. Mech.*, V. 2, pp. 1-17.

Chang, T. Y.; Taniguchi, H.; and Chen, W. F., 1987, "Nonlinear Finite Element Analysis of Reinforced Concrete Panels," *J. Struct. Engrg.*, ASCE, V. 113, No. 1, Jan., pp. 122-140.

Channakeshava, C., and Iyengar, K. T. S. R., 1988, "Elasto-Plastic Cracking Analysis of Reinforced Concrete," *J. Struct. Engrg.*, ASCE, V. 114, No. 11, Nov., pp. 2421-2438.

Chen, W. F., 1982, *Plasticity in Reinforced Concrete*, McGraw-Hill, New York.

Clauss, D. B., 1987, "Round-Robin Pretest Analyses of a 1:6-Scale Reinforced Concrete Containment Model Subject to Static Internal Pressurization," *Report* SAND87-0891, May, Sandia Natl. Labs., Albuquerque, N.M.

Clough, R. W., 1962, "The Stress Distribution of Norfolk Dam," Series 100, Issue 19, Institute of Engineering Research, University of California, Berkeley, Aug.

Colville, J., and Abbasi, J., 1974, "Plane Stress Reinforced Concrete Finite Elements," *J. Struct. Div.*, ASCE, V. 100, No. ST5, May, pp. 1067-1083.

Computational Modelling of Concrete Structures, 1994, H. Mang, N. Bicanic, and R. deBorst, eds., Pineridge Press, London.

Computer-Aided Analysis and Design of Concrete Structures, 1984, F. Damjanic et al., eds., Pineridge Press, London.

Computer-Aided Analysis and Design of Concrete Structures, 1990, edited by N. Bicanic and H. Mang, Pineridge Press, London.

Concrete Design Based on Fracture Mechanics, 1992, edited by W. H. Gerstle and Z. P Bazant, ACI SP-134, American Concrete Institute, Detroit.

Cosserat, E., and Cosserat, F., 1909, *Theorie des Corps Deformables*, Hermann, Paris.

Crisfield, M. A., and Wills, J., 1989, "Analysis of R/C Panels Using Different Concrete Models," *J. Engrg. Mech.*, V. 115, No. 3, Mar., pp. 578-597.

Dahlblom, O., and Ottosen, N., 1990, “Smeared Crack Analysis Using a Generalized Fictitious Crack Model,” *J. Engrg. Mech.*, ASCE, V. 116, No. 1, Jan., pp. 55-76.

Dam Fracture, 1991, V. Saouma, R. Dunger and D. Morris, eds., Electrical Power Research Inst., Palo Alto.

Dam Fracture and Fatigue, 1994, E. Bourdarot, J. Mazars, and V. Saouma, eds., Balkema, Rotterdam.

Darwin, D., 1985, “Concrete Crack Propagation-Study of Model Parameters,” *Finite Element Analysis of Reinforced Concrete Structures*, edited by C. Meyer and H. Okamura, ASCE, New York, pp. 184-203.

Darwin, D., 1993, “Reinforced Concrete,” Chapter 4, *Finite Element Analysis of Reinforced Concrete Structures II*, edited by J. Isenberg, American Society of Civil Engineers, pp. 203-232.

Darwin, D., and Pecknold, D. A. W., 1976, “Analysis of RC Shear Panels Under Cyclic Loading,” *J. Struct. Div.*, ASCE, V. 102, No. ST2, Feb., pp. 355-369.

Darwin, D.; McCabe, S. L.; Brown, C. J.; and Tholen, M. L., 1994, “Fracture Analysis of Steel-Concrete Bond,” *Fracture and Damage in Quasibrittle Structures: Experiment, Modeling, and Computer Analysis*, Z. P. Bazant, Z. Bittnar, M. Jirasek and J. Mazars, eds., E&FN Spon, London, pp. 549-556.

De Borst, R., and Nauta, P., 1985, “Nonorthogonal Cracks in a Smeared Finite Element Model,” *Engrg. Comp.*, V. 2, No. 1, pp. 35-46.

De Borst, R., 1986, “Nonlinear Analysis of Frictional Materials,” PhD Thesis, Technical University, Delft, Netherlands.

Desai, C. S.; Zaman, M. M.; Lightner, J. G.; and Siriwardane, H. J., 1984, “Thin-Layer Element for Interfaces and Joints,” *Int. J. Numer. Anal. Meth. Geotech.*, V. 8., pp. 19-43.

Dodds, R. H.; Darwin, D.; and Leibengood, L. D., 1984, “Stress Controlled Smeared Cracking in R/C Beams,” *J. Struct. Engrg.*, ASCE, V. 110, No. 9, Sept., pp. 1959-1976.

Droz, P., and Bazant, Z. P., 1989, “Nonlocal Analysis of Stable States and Stable Paths of Propagation of Damage Shear Bands,” *Cracking and Damage*, edited by J. Mazars and Z. P. Bazant, Elsevier, London, pp. 415-425.

Du, J.; Kobaysh, A. S.; and Hawkins, N. M., 1990, “Direct FEM Analysis of Concrete Fracture Specimens,” *J. Engrg. Mech*., ASCE, V. 116, No. 3, Mar., pp. 605-619.

Dvorkin, E.; Cuitino, A.; and Gioia, G., 1990, “Finite Elements with Displacement Interpolated Embedded Localization,” *Int. J. Num. Meth. in Engrg*., V. 30, pp. 541-564.

Elfgren, L., 1989, *Fracture Mechanics of Concrete Structures*, RILEM Report, Chapman and Hall, London.

Elices, M., and Planas, J., 1989, “Material Models,” Chapter 3 in *Fracture Mechanics of Concrete Structures*, edited by L. Elfgren, Chapman and Hall, London, pp. 16-66.

Elfgren, L., 1992, “Round Robin Analysis and Tests of Anchor Bolts,” *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 865-869.

Eligehausen, R., and Ozbolt, J., 1990, “Size Effect in Anchorage Behavior,” *Proc.*, 8th Biennial European Conf. on Fracture, ECF8, Oct. 1-5, Torino, pp. 2671-2677, EMAS, U.K.

Eligehausen, R., and Ozbolt, J., 1992, “Influence of Crack Width on the Concrete Cone Failure Load,” *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 876-881.

Erdogan, F., and Sih, G. C., 1963, “On the Crack Extension in Plates Under Plane Loading and Transverse Shear,” *J. Basic Engineering*, ASME, V. 85, pp. 519-527.

Eringen, A. C., 1965, “Theory of Micropolar Continuum,” *Proceedings*, 9th Midwestern Mechanics Conf., Univ. of Wisconsin, Madison, pp. 23-40.

Eringen, A. C., 1966, “A Unified Theory of Thermomechanical Materials,” *Int. J. Engrg. Sci*, V. 4, pp. 179-202.

Feenstra, P. H.; de Borst, R.; and Rots, J. G., 1991a, “Numerical Study on Crack Dilatancy. I: Models and Stability Analysis,” *J. Engrg. Mech.*, ASCE, V. 117, Apr., No. 4, pp. 733-753.

Feenstra, P. H.; de Borst, R.; and Rots, J. G., 1991b, “Numerical Study on Crack Dilatancy. II: Applications,” *J. Engrg. Mech.*, ASCE, V. 117, No. 4, Apr., pp. 754-769.

Finite Element Analysis of Reinforced Concrete II, 1993, edited by J. Isenberg, American Society of Civil Engineers, New York, 544 pp.

Firrao, D., Oct., 1990, “Fracture Behaviour and Design of Materials and Structures,” *Proc.*, 8th Biennial European Conf. on Fracture, ECF8, Torino, Italy.

Fish, J., and Belytschko, T., 1988, “Elements with Embedded Localization Zones,” *J. Comp. and Structures*, V. 30, pp. 247-256.

Fracture and Damage in Quasibrittle Structures: Experiment, Modeling, and Computer Analysis, 1994, Z. P. Bazant, Z. Bittnar, M. Jirasek and J. Mazars, eds., E&FN Spon, London.

Fracture Mechanics: Application to Concrete, 1989, edited by V. C. Li and Z. P Bazant, ACI SP-118, American Concrete Institute, Detroit.

Fracture Mechanics of Concrete Structures, 1992, Proc. 1st Int. Conf. Fract. Mech. of Conc. Struct. (FraMCoS-1), edited by Z. P. Bazant, Elsevier, London.

Fracture Mechanics of Concrete Structures, 1995, Proc. 2nd Int. Conf. Fract. Mech. of Conc. and Conc. Struct. (FraMCoS-2), edited by F. H. Wittmann, Aedificatio Publications, Freiburg, Germany.

Fracture of Concrete and Rock, 1987, edited by S. P. Shah and S. E. Swartz, Proc. of the SEM-RILEM International Conference, June 17-19, 1987, Houston.

Gajer, G., and Dux, P. F., 1990, “Crack Band Based Model for FEM Analysis of Concrete Structures,” *J. Struct. Engrg.*, ASCE, V. 116, No. 6, June, pp. 1696-1714.

Gajer, G., and Dux, P. F., 1991, “Simplified Nonorthogonal Crack Model for Concrete,” *J. Struct. Engrg*., ASCE, V. 117, No. 1, Jan., pp. 149-164.

Garcia-Alvarez, V. O.; Carol., I.; and Gettu, R., 1994, “Numerical Simulation of Fracture in Concrete Using Joint Element,” *Anales de Mecanica de la Fractura*, V. 11, pp. 75-80.

Gens, A.; Carol, I.; and Alonso, E., 1988, “An Interface Element Formulation for the Analysis of Soil-Reinforcement Interaction,” *Computers and Geotechnics*, V. 7, pp. 133-151.

Gerstle, K. H., July, 1987, “Strength Design of Reinforced-Concrete Hydraulic Structures,” *Technical Report* SL-80-4, US Army Corps of Engineers.

Gerstle, W. H., 1982, “The Fracture Mechanics of Bond in Reinforced Concrete,” MS Thesis, Dept. of Struct. Engrg., Cornell Univ.

Gerstle, W. H., 1986, “Finite and Boundary Element Modelling of Crack Propagation in Two- and Three-Dimensions Using Interactive Computer Graphics,” Dept. of Struct. Engrg., Cornell Univ., Ithaca, N.Y.

Gerstle, W. H.; Martha, L. F.; and Ingraffea, A. R., 1987, “Finite and Boundary Element Modelling of Crack Propagation in Two- and Three-Dimensions,” *Engrg. with Comp*., V. 2, pp. 167-183.

Gerstle, W. H., and Abdalla, J. E., Jr., 1990, “Finite Element Meshing Criteria for Crack Problems,” *Fracture Mechanics: Twenty-First Symposium*, ASTM, STP 1074, American Society for Testing and Materials, Philadelphia, pp. 509-521.

Gerstle, W. H., and Xie, M., 1992, “FEM Modeling of Fictitious Crack Propagation in Concrete,” *J. Engrg. Mech.*, ASCE, V. 118, Feb., No. 2, pp. 416-434.

Goodman, R. E.; Taylor, R. L.; and Brekke, T. L., 1968, “A Model for the Mechanics of Jointed Rock,” *J. Soil Mech. and Found. Div*., ASCE, V. 94, No. 3, pp. 637-659.

Gopalaratnam, V. S., and Ye, B. S., 1991, “Numerical Characterization of Nonlinear Fracture Process in Concrete,” *Engrg. Fract. Mech*., V. 40, No. 6, pp. 991-1006.

Griffith, A. A., 1920, “The Phenomenon of Rupture and Flow in Solids,” *Philosophical Transactions*, Royal Society of London, Series A, V. 221, pp. 163-198.

Gupta, A. K., and Maestrini, S. R., 1989a, “Post-Cracking Behavior of Membrane Reinforced Concrete Elements Including Tension Stiffening,” *J. Struct. Engrg.*, ASCE, V. 115, No. 4, Apr., pp. 957-976.

Gupta, A. K., and Maestrini, S. R., 1989b, “Unified Approach to Modeling Post-Cracking Membrane Behavior of Reinforced Concrete,” *J. Struct. Engrg.*, ASCE, V. 115, No. 4, Apr., pp. 977-993.

Gustafsson, P. J., and Hillerborg, A., 1984, “Improvements in Concrete Design Achieved through the Application of Fracture Mechanics,” *Application of Fracture Mechanics to Cementitious Materials*, NATO Advanced Research Workshop, Northwestern University, Evanston, Ill.

Gustafsson, P. J., 1985, *Fracture Mechanics Studies of Non-Yielding Materials Like Concrete*, TVBM-1007, Division of Building Materials, University of Lund, Sweden.

Gustafsson, P. J., and Hillerborg, A., 1988, “Sensitivity in Shear Strength of Longitudinally Reinforced Concrete Beams to Fracture Energy of Concrete,” *ACI Struct. J.*, V. 85, No. 5, May, pp. 286-294.

Hand, F. R.; Pecknold, D. A.; and Schnobrich, W. C., 1973, “Nonlinear Layered Analysis of RC Plates and Shells,” *J. Struct. Div*., ASCE, V. 99, No. ST7, July, pp. 1491-1505.

Hellen, T. K., 1975. “On the Method of Virtual Crack Extensions,” *Int. J. Num. Meth. in Engrg*., V. 9, pp. 187-207.

Hellier, A. K.; Sansalone, M.; Ingraffea, A. R.; Carino, N. J.; and Stone, W. C., 1987, “Finite Element Analysis of the Pullout Test Using a Nonlinear

Discrete Cracking Approach," *Cement, Concrete and Aggregates*, V. 9, No. 1, pp. 20-29.

Henshell, R. D., and Shaw, K. G., 1975, "Crack Tip Finite Elements are Unnecessary," *Int. J. Num. Meth. in Engrg.*, V. 9, pp. 495-507.

Heuze, F. E., and Barbour, T. G., 1982, "New Models for Rock Joints and Interfaces," *J. Geotech. Engrg. Div.*, ASCE, V. 108, No. GT5, pp. 757-776.

Hillerborg, A.; Modeer, M.; and Petersson, P. E., 1976, "Analysis of Crack Formation and Crack Growth in Concrete by Means of Fracture Mechanics and Finite Elements," *Cem. and Conc. Res*., V. 6, 773-782.

Hillerborg, A., and Rots, J., 1989, "Crack Opening and Numerical Modelling," *Fracture Mechanics of Concrete Structures*, L. Elfgren, ed., Chapman and Hall, London, pp. 128-146.

Hohberg, J., 1990, "A Note on Spurious Oscillations in FEM Joint Elements," *Earthquake Engrg. and Struct. Dyn*., V. 19, pp. 773-779.

Hohberg, J., 1992a, "A Joint Element for the Nonlinear Dynamic Analysis of Arch Dams," *Technical Report* 186, Institute of Structural Engineering, ETH Zurich.

Hohberg, J., 1992b, "Multimechanism Plasticity with Coupled Damage in Tension and Shear," *Computational Plasticity* (COMPLAS III), D. Owen, E. Onate, and E. Hinton, eds., V. 2, Pineridge Press, London, pp. 1503-1514.

Hu, H. T., and Schnobrich, W. C., 1991, "Nonlinear Finite Element Analysis of Reinforced Concrete Plates and Shells under Monotonic Loading," *Computers and Structures*, V. 38, No. 5/6, pp. 637-651.

Hussain, M. A.; Pu, S. L.; and Underwood, J. H., 1974, "Strain Energy Release Rate for a Crack Under Combined Mode I and Mode II," *Fracture Analysis*, ASTM, STP 560, American Society for Testing and Materials, pp. 2-28.

Ingraffea, A. R., 1977, "Discrete Fracture Propagation in Rock: Laboratory Tests and Finite Element Analysis," PhD Dissertation, University of Colorado, Boulder.

Ingraffea, A. R., and Manu, C., 1980, "Stress-Intensity Factor Computation in Three Dimensions with Quarter-Point Elements," *Int. J. Num. Meth. in Engrg*., V. 15, pp. 1427-1445.

Ingraffea, A. R., and Chappell, J. F., 1981, "A Fracture Mechanics Investigation of the Cracking of Fontana Dam," *Technical Report* No. 81-7, Cornell University, Ithaca, NY.

Ingraffea, A. R., 1983, "Numerical Modelling of Crack Propagation," *Rock Fracture Mechanics*, Springer-Verlag, Wien-New York, pp. 151-208.

Ingraffea, A. R.; Gerstle, W. H.; Gergely, P.; and Saouma, V., 1984, "Fracture Mechanics of Bond in Reinforced Concrete," *J. Struct. Engrg.*, ASCE, V. 110, No. 4, Apr., pp. 871-890.

Ingraffea, A. R., and Saouma, V., 1984, "Numerical Modeling of Discrete Crack Propagation in Reinforced and Plain Concrete," *Fracture Mechanics of Concrete: Structural Application and Numerical Calculation*, edited by G. Sih and A. DiTommaso, Martinus Nijhoff, Hingham, Mass., pp. 171-225.

Ingraffea, A. R., and Gerstle, W. H., 1985, "Non-Linear Fracture Models for Discrete Crack Propagation," *Application of Fracture Mechanics to Cementitious Composites*, NATO-ARW, edited by S. P. Shah, September 4-7, 1984, Martinus-Nijhoff, pp. 247-285.

Ingraffea, A. R., and Panthaki, M., 1986, "Analysis of Shear Fracture Tests of Concrete Beams," *Finite Element Analysis of Reinforced Concrete Structures*, edited by C. Meyer and H. Okamura, American Society of Civil Engineers, New York, pp. 151-173.

Ingraffea, A. R., 1990, "Case Studies of Simulation of Fracture in Concrete Dams," *Engrg. Fracture Mechanics*, V. 35, No. 1/2/3, pp. 553-564.

Jenq Y. S., and Shah, S. P., 1985, "Nonlinear Fracture Parameters for Cement Based Composites: Theory and Experiments," *Application of Fracture Mechanics to Cementitious Composites*, Proc. NATO Adv. Res. Workshop, V. 15, pp. 319-360.

Jirasek, M., and Bazant, Z. P., 1994, "Localizaton Analysis of Nonlocal Model Based on Crack Interactions," *J. Engrg. Mech.*, ASCE, V. 120, No. 3, Mar., pp. 1521-1542.

Jirasek, M., and Bazant, Z. P., 1995a, "Macroscopic Fracture Characteristics of Random Particle Systems," *Int. J. Fracture*, V. 69, pp. 201-228.

Jirasek, M., and Bazant, Z. P., 1995b, "Particle Model for Quasibrittle Fracture," *J. Engrg. Mech*., ASCE, V. 121, No. 9, Sept., pp. 1016-1025.

Jordan, W. B., 1970, "The Plane Isoparametric Structural Element," General Electric Co., Knolls Atomic Power Lab, *Report* No. KAPL-M-7112.

Kaplan, M. F., 1961, "Crack Propagation and the Fracture of Concrete," ACI JOURNAL, *Proceedings* V. 58, No. 5, May, pp. 591-610.

Karsan, I. D., and Jirsa, J. O., 1969, "Behaviour of Concrete Under Compressive Loadings," *J. Struct Engrg.*, ASCE, V. 95, No. ST12, Dec., pp. 2543-2563.

Kay, G. J.; Govindjee, S.; Laursen, T.; and Maker, B., 1992, "Development of Concrete Tensile Fracture and Concrete/Rebar Bond Strength Models," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 849-852.

Kesler, C.; Naus, D.; and Lott, J., 1972, "Fracture Mechanics—Its Applicability to Concrete," *Proc.*, 1971 Int. Conf. on Mechanical Behavior of Materials, V. IV, pp. 113-124.

Keuser, M., and Mehlhorn, G., 1987, "Finite Element Models for Bond Problems," *J. Struct. Engrg.*, ASCE, V. 113, No. 10, Oct., pp. 2160-2173.

Klisinski, M.; Runesson, K.; and Sture, S., 1991, "Finite Element with Inner Softening Band," *J. Engrg. Mech.*, ASCE, V. 117, No. 3, Mar., pp. 575-587.

Leibengood, L. D.; Darwin, D.; and Dodds, R. H., 1986, "Parameters Affecting FE Analysis of Concrete Structures," *J. Struct. Engrg.*, ASCE, V. 112, No. 2, Feb., pp. 326-341.

Lewinski, P. M., and Wojewodzki, W., 1991, "Integrated Finite Element Model for Reinforced Concrete Slabs," *J. Struct. Engrg.*, ASCE, V. 117, No. 4, Apr., pp. 1017-1038.

L'Hermite, R. et al., 1952, "Déformation du Béton," *Batir*, V. 27, pp. 24-28.

Li, V. C., and Liang, E., 1986, "Fracture Processes in Concrete and Fiber Reinforced Cementitious Composites," *J. Engrg. Mech*., ASCE, V. 112, No. 6, June, pp. 566-586.

Li, Y. N., and Liang, R. Y., 1992, "Stability Theory of Cohesive Crack Model," *J. Engrg. Mech*., ASCE, V. 118, No. 3, March, pp. 587-603.

Li, Y.-N., and Bazant, Z. P., 1994, "Eigenvalue Analysis of Size Effect for Cohesive Crack Model," *Int. J. Fracture*, V. 66, No. 4, pp. 213-226.

Liang, R. Y., and Li, Y.-N., 1991, "Study of Size Effect in Concrete Using Fictitious Crack Model," *J. Engrg. Mech.*, ASCE, V. 117, No. 7, July, pp. 1631-1651.

Liaw, B. M.; Jeang, F. L.; Hawkins, N. M.; and Kobayashi, A. S., 1990a, "Fracture-Process Zone for Mixed-Mode Loading of Concrete," *J. Engrg. Mech.*, ASCE, V. 116, No. 7, July, pp. 1560-1579.

Liaw, B. M.; Jeang, F. L.; Du, J. J.; Hawkins, N. M.; and Kobayashi, A. S., 1990b, "Improved Nonlinear Model for Concrete Fracture," *J. Engrg. Mech.*, ASCE, V. 116, No. 2, Feb., pp. 429-445.

Lin, C. S., and Scordelis, A. C., 1975, "Nonlinear Analysis of RC Shells of General Form," *J. Struct. Div.*, ASCE, V. 101, No. ST3, Mar., pp. 523-538.

Linsbauer, H. N.; Ingraffea, A. R.; Rossmanith, H. P.; and Wawrzynek, P. A., 1989a, "Simulation of Cracking in Large Arch Dam: Part I," *J. Struct. Div.*, ASCE, V. 115, No. 7, July, pp. 1599-1615.

Linsbauer, H. N.; Ingraffea, A. R.; Rossmanith, H. P.; and Wawrzynek, P. A., 1989b, "Simulation of Cracking in Large Arch Dam: Part II," *J. Struct. Div.*, ASCE, V. 115, No. 7, July, pp. 1616-1630.

Lopez, C. M., and Carol, I., 1995, "Fracture Analysis of Concrete Microstructure Using Interface Elements," *Anales de Mecanica de la Fractura*, V. 12, pp. 75-80.

Lotfi, H. R., 1992, "Finite Element Analysis of Fracture of Concrete and Masonry Structures," PhD Thesis, Dept. of CEAE, University of Colorado, Boulder.

Lotfi, H. R., and Shing, P. B., 1994, "Interface Model Applied to Fracture of Masonry Structures," *J. Engrg. Mech.*, ASCE, V. 120, No. 1, Jan., pp. 63-80.

Lotfi, H. R., and Shing, P. B., 1995, "Embedded Representation of Fracture in Concrete with Mixed Finite Elements," *Int. J. Num. Meth. in Engrg*, in press.

Ma, T.; Niwa, J.; and McCabe, S. L., 1992, "Evaluation of the Shear Behavior of Reinforced Concrete Beams—A Comparison of Code and Fracture Mechanics Predictions," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 788-793.

Malvar, L. J., 1992, "Mixed Moded Fracture in Concrete," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 677-688.

Martha, L., 1989, "Topological and Geometrical Modeling Approach to Numerical Discretization and Arbitrary Fracture Simulation in Three-Dimensions," PhD Dissertation, School of Civil Engineering, Cornell University, Ithaca.

Martha, L. F.; Llorca, J.; Ingraffea, A. R.; and Elices, M., 1991, "Numerical Simulation of Crack Initiation and Propagation in an Arch Dam," *Dam Engineering*, V. 2, No. 3, pp. 193-214.

Massicote, B.; Elwi, A. E.; and MacGregor, J. G., 1990, "Tension-Stiffening Model for Planar Reinforced Concrete Members," *J. Struct. Engrg.*, ASCE, V. 116, No. 11, Nov., pp. 3039-3058.

Mau, S. T., and Yang, M. S., 1977, "Some Applications of a Hybrid Finite Element Method to Crack Problems," *Fracture Mechanics and Technology*, edited by George Sih, Noordhoff, Netherlands, V. 11, pp. 1467-1482.

Mehlhorn G.; Herrmann, R.; and Zerna, W., eds., 1978, "Nonlinear Behaviour of Reinforced Concrete Spatial Structures," *Proc.*, Int. Assoc. of Shell Structures, July 3-7, Darmstadt, Werner-Verlag, Dusseldorf, Germany.

Meyer, C.; and Bathe, K.-J., 1982, "Nonlinear Analysis of R/C Structures in Practice," *J. Struct. Div.*, ASCE, V. 108, No. ST7, July, pp. 1605-1622.

Modeer, M., 1979, "A Fracture Mechanics Approach to Failure Analyses of Concrete Materials," *Report* TVBM-1001, thesis, Div. of Build. Mat., Univ. of Lund, Sweden.

Nam, C.-H., and Salmon, C. G., 1974, "Finite Element Analysis of Concrete Beams," *J. Struct. Div.*, ASCE, V. 100, No. ST12, Dec., pp. 2419-2432.

Ngo, D., and Scordelis, A. C., 1967, "Finite Element Analysis of Reinforced Concrete Beams," ACI JOURNAL, *Proceedings* V. 64, No. 3, Mar., pp. 152-163.

Nilson, A. H., 1967, "Finite Element Analysis of Reinforced Concrete," PhD Thesis, Dept. of Civil Engineering, University of California, Berkeley, CA.

Nilson, A. H., 1968, "Nonlinear Analysis of Reinforced Concrete by Finite Element Method," ACI JOURNAL, *Proceedings*, V. 65, No. 9, Sept., pp. 757-766.

Ortiz, M.; Leroy, Y.; and Needleman, A., 1987, "A Finite Element Method for Localized Failure Analyis," *J. Comp. Meth in Appl Mech. and Engrg.*, V. 61, pp. 189-214.

Ozbolt, J., and Eligehausen, R., 1991, "Analysis of Reinforced Concrete Beams without Shear Reinforcement Using Non-Local Microplane Model," *Fracture Processes in Concrete, Rock and Ceramics*, RILEM, pp. 919-930.

Ozbolt, J., and Eligehausen, R., 1992, "Simulation of Cyclic Bond-Slip Behavior," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 912-917.

Ozbolt, J., and Bazant, Z. P., 1992, "Microplane Model for Cyclic Triaxial Behavior of Concrete," *J. Engrg. Mech.*, ASCE, V. 118, No. 7, July, pp. 1365-1386.

Ozbolt, J., and Bazant, Z. P., 1996, "Numerical Smeared Fracture Analysis: Nonlocal Microcrack Interaction Approach," *Int. J. Num. Meth. Engrg.*, V. 39, pp. 635-661.

Pagnoni, T.; Slater, J.; Ameur-Moussa, R.; and Buyukozturk, O., 1992, "A Nonlinear Three-Dimensional Analysis of Reinforced Concrete Based on a Bounding Surface Model," *Computers and Structures*, V. 43, No. 1, pp. 1-12.

Papadrakis, M., 1981, "A Method for the Automatic Evaluation of the Dynamic Relaxation Parameters," *Comp. Meth. Appl. Mech. Engrg.*, V. 25, No. 1, pp. 35-48.

Parks, D. M., 1974, "A Stiffness Derivative Finite Element Technique for Determination of Crack Tip Stress Intensity Factors," *Int. J. of Fract.*, V. 10, No. 4, pp. 487-502.

Peruchio, R.; Saxena, M.; and Kela, A., 1989, "Automatic Mesh Generation from Solid Models Based on Recursive Spatial Decompositions," *Int. J. Num. Methods in Engrg.*, V. 28, pp. 2469-2501.

Petersson, P.-E., and Gustafsson, P. J., 1980, "A Model for Calculation of Crack Growth in Concrete-Like Materials," *Numerical Methods in Fracture Mechanics*, edited by Owen and Luxmoore, Pineridge Press, Swansea, U.K., pp. 707-719.

Petersson, P. E., 1981, "Crack Growth and Development of Fracture Zones in Plain Concrete and Similar Materials," *Report* TVBM-1006, Division of Building Materials, Lund Institute of Technology, Sweden.

Planas, J., and Elices, M., 1991, "Nonlinear Fracture of Cohesive Materials," *Int. J. Fracture*, V. 51, No. 2, pp. 139-158.

Planas, J., and Elices, M., 1992, "Asymptotic Analysis of a Cohesive Crack: 1. Theoretical Background," *Int. J. Fracture*, V. 55, No. 2, pp. 153-178.

Planas, J., and Elices, M., 1993a, "The Equivalent Elastic Crack: 1. Load-Y Equivalences," *Int. J. Fracture*, V. 61, No. 2, pp. 159-172.

Planas, J., and Elices, M., 1993b, "Asymptotic Analysis of a Cohesive Crack: 2. Influence of the Softening Curve," *Int. J. Fracture*, V. 64, No. 3, pp. 221-237.

Pramono, E., and Willam, K, 1987, "Fracture Energy-Based Plasticity Formulation of Plain Concrete," *J. Engrg. Mech.*, ASCE, V. 115, No. 6, June, pp. 1183-1204.

Pukl, R.; Schlottke, B.; Ozbolt, J.; and Eligehausen, R., 1992, "Computer Simulation: Splitting Tests of Concrete Thick-Walled Rings," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London.

Rahulkumar, P., 1992, "FEM Modeling of Crack Propagation in Concrete Using an Energy Approach," MS Thesis, Dept. of Civil Engineering, Univ. of New Mexico, Albuquerque.

Rashid, Y. R., 1968, "Ultimate Strength Analysis of Prestressed Concrete Pressure Vessels," *Nuclear Engineering and Design*, V. 7, pp. 334-344.

Rice, J. R., 1968, "A Path Independent Integral and the Approximate Analysis of Strain Concentrations by Notches and Cracks," *J. Appl. Mech.*, V. 35, pp. 379-386.

Roelfstra, P. E.; Sadouki, H.; and Wittmann, F. H., 1985, "Le Beton Numerique," *Mater. Struct.*, RILEM, V. 18, pp. 309-317.

Rots, J., 1988, "Computational Modelling of Concrete Fracture," PhD Thesis, Delft University of Technology, the Netherlands.

Rots, J. G.; Kusters, G. M. A.; and Blaawendraad, J., 1984, "The Need for Fracture Mechanics Options in Finite Element Models for Concrete Structures," *Computer-Aided Analysis and Design of Concrete Structures*, Part I, pp. 19-32.

Rots, J. G., and de Borst, R., 1987, "Analysis of Mixed-Mode Fracture in Concrete," *J. Engrg. Mech.*, V. 113, No. 11, Nov., pp. 1739-1758.

Rots, J., and Schellekens, J., 1990, "Interface Elements in Concrete Mechanics," edited by N. Bicanic and H. Mang, *Computer Aided Analysis and Design of Concrete Structures*, Pineridge Press, London, V. 2, pp. 909-918.

Salah El-Din, A. S., and El-Adawy Nassef, M. M., 1975, "A Modified Approach for Estimating the Cracking Moment of Reinforced Concrete Beams," ACI JOURNAL, *Proceedings* V. 72, No. 7, July, pp. 356-360.

Saouma, V. E., 1981, "Interactive Finite Element Analysis of Reinforced Concrete: A Fracture Mechanics Approach," PhD Thesis, Dept. of Struct. Engrg., Cornell Univ., Ithaca.

Saouma, V. E., and Schwemmer. D., 1984, "Numerical Evaluation of the Quarter-Point Crack Tip Element," *Int. J. Num. Meth. in Engrg.*, V. 20, pp. 1629-1641.

Saouma, V. E.; Ayari, M. L.; and Boggs, H., 1987, "Fracture Mechanics of Concrete Gravity Dams," *Proceedings*, SEM-RILEM Int. Conf. of Fracture and Concrete Rock, Houston, June 17-19, pp. 496-519.

Sapadis, N., and Perucchio, R., 1989, "Advanced Techniques for Automatic Finite Element Meshing from Solid Models," *Computer-Aided Design*, V. 21, pp. 248-253.

Schellekens, J., and de Borst, R., 1993, "On the Numerical Integration of Interface Elements," *Int. J. Num. Meth. Engrg.*, V. 36, pp. 43-66.

Schlangen, E., and van Mier, J. G. M., 1992, "Shear Fracture in Cementitious Composites Part II: Numerical Simulations," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 671-676.

Schnobrich, W. C., 1977, "Behavior of Reinforced Concrete Structures Predicted by the Finite Element Method," *Computers and Structures*, V. 7, pp. 265-376.

Schreyer, H. L., 1990, "Analytical Solutions for Nonlinear Strain-Gradient Softening and Localization," *J. Appl. Mech.*, V. 57, pp. 522-528.

Schroeder, W. J., and Shepard, M. S., 1988, "Geometry-based Fully Automatic Mesh Generation and the Delauney Triangulation," *Int. J. Num. Meth. in Engrg.*, V. 26, pp. 2503-2515.

Sha, G. T., and Yang, C.-T., 1990, "Use of Analytically Decoupled Near-Tip Displacement Solutions for Calculating the Decoupled Weight Functions of Mixed Fracture Modes," *Engrg. Fract. Mech.*, V. 37, No. 3, pp. 591-610.

Shepard, M. S., Oct., 1984, "Automated Analysis Model Generation," *Computer Aided Design in Civil Engineering*, American Society of Civil Engineers, New York, pp. 92-99.

Shih, C. F.; deLorenzi, H. G.; and German, M. D., 1976, "Crack Extension Modelling with Singular Quadratic Isoparametric Elements," *Int. J. Fract.*, V. 12, pp. 647-651.

Shivakumar, K. N.; Tan, P. W.; and Newman, J. C., 1988, "A Virtual Crack-Closure Technique for Calculating Stress Intensity Factors for Cracked Three-Dimensional Bodies," *Int. J. Fracture*, V. 36, pp. R43-R50.

Shumin, C., and Xing, Z., 1990, "Generalized Stiffness Derivative Method for Mixed Mode Crack Problems," *Mechanics Res. Comm.*, V. 17, No. 6, pp. 437-444.

Sih, G. C., 1974, "Strain Energy Density Factor Applied to Mixed-Mode Crack Problems," *Int. J. of Fracture*, V. 10, pp. 305-321.

Size Effect in Concrete Structures, 1994, H. Mihashi, H. Okamma, and Z. P. Bazant, eds., E & F N Spon, London.

Skrikerud, P., 1986, "Discrete Crack Modelling for Dynamically Loaded, Unreinforced Concrete Structures," *Earthquake Engrg. and Struct. Dyn.*, V. 14, pp. 297-315.

Sluys, L. J., and de Borst, R., 1992, "Analysis of Impact Fracture in a Double-Notched Specimen Including Rate Effects," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 610-615.

Sousa, J. L.; Martha, L. F.; Wawrzynek, P. A.; and Ingraffea, A. R., 1989, *Simulation of Non-Planar Crack Propagation in Three-Dimensional Structures in Concrete and Rock Fracture of Concrete and Rock Recent Develop-*

ments, S.P. Shah, S. E. Swartz, and B. I. G. Barr, eds., Elsevier Science Publishing, New York, pp. 254-264.

Stankowski, T., 1990, "Numerical Simulation of Progressive Failure in Particle Composites," PhD Thesis, Dept. of CEAE, Univ. of Colorado, Boulder.

Stankowski, T.; Etse, G.; Runesson, K.; Sture, S.; and Willam, K., 1992, "Composite Analysis with Discrete and Smeared Crack Concepts," *Fracture Mechanics of Concrete Structures*, edited by Z. P. Bazant, Elsevier, London, pp. 269-274.

Stankowski, T.; Runesson, K.; and Sture, S., 1993, "Fracture and Slip of Interfaces in Cementitious Composites. I: Characteristics, II: Implementation," *J. Engrg. Mech*. ASCE, V. 119, No. 2, Feb., pp. 292-327.

Struman, G. M.; Shah, S. P.; and Winter, G., 1965, "Effect of Flexural Strain Gradients on Microcracking and Stress-Strain Behavior of Concrete," ACI JOURNAL, *Proceedings* V. 62, No. 7, July, pp. 805-822.

Swenson, D. V., 1986, "Modeling Mixed-Mode Dynamic Crack Propagation Using Finite Elements," PhD Dissertation, School of Civil and Environmental Engineering, Cornell University, Ithaca.

Swenson, D. V., and Ingraffea, A. R., 1991, "The Collapse of the Schoharie Creek Bridge: A Case Study in Concrete Fracture Mechanics," *Int. J. of Fracture*, V. 51, pp. 73-92.

Swenson, D. V., and Ingraffea, A. R., 1988, "Modeling Mixed-Mode Dynamic Crack Propagation Using Finite Elements: Theory and Applications," *Comput. Mech.*, V. 3, pp. 381-397.

Tada, H.; Paris, P. C.; and Irwin, G. R., 1973, *The Stress Analysis of Cracks Handbook*, Del Research Corp., Hellertown.

Tasdemir, M. A.; Maji, A. K.; and Shah, S. P., 1990, "Crack Propagation in Concrete under Compression," *J. Engrg. Mech.*, ASCE, V. 116, No. 5, May, pp. 1058-1077.

Tong, P., and Pian, T. H. H., 1973, "On the Convergence of the Finite Element Method for Problems with Singularity," *Int. J. Solids Structures*, V. 9, pp. 313-321.

Tong, P.; Pian, T. H. H.; and Lasry, S. J., 1973, "A Hybrid Element Approach to Crack Problems in Plane Elasticity," *Int. J. Num. Meth. in Engrg.*, V. 7, pp. 293-308.

Tracey, D. M., 1971, "Finite Elements for the Determination of Crack Tip Elastic Stress Intensity Factors," *Engrg. Fract. Mech.*, V. 3, pp. 255-265.

Underwood, P., 1983, "Dynamic Relaxation," *Computational Methods in Mechanics for Transient Analysis*, edited by T. Belytschko and T. J. R. Hughes, North-Holland Publishing, Amsterdam, pp. 245-265.

Valliappan, S., and Doolan, T., 1972, "Nonlinear Stress Analysis of Reinforced Concrete," *J. Struct. Div.*, ASCE, V. 98, No. ST4, Apr., pp. 885-898.

Van Mier, J. G. M.; Rots, J. G.; and Bakker, A., 1991, "Fracture Processes in Concrete, Rock and Ceramics," *Proc.*, Intl. RILEM/ESIS Conf., June, Noordwijk, The Netherlands.

Van Mier, J. G. M., and Vonk, R., 1991, "Fracture of Concrete and Multiaxial Stress-Recent Developments," *Mater. Struct.*, RILEM, V. 24, pp. 61-65.

Vecchio, F. J., 1986, "Nonlinear Finite Element Analysis of Reinforced Concrete Membranes," *ACI Struct. J.*, V. 86, No. 1, Jan.-Feb., pp. 26-35.

Vecchio, F. J., and Collins, M. P., 1982, "The Response of Reinforced Concrete to Inplane Shear and Normal Stress," *Publication* 82-03, Dept. of Civil Engrg., Univ. of Toronto.

Vonk, R., 1992, "Softening of Concrete Loaded in Compression," PhD Thesis, Technische Univeriteit Eindhoven, Eindhoven, the Netherlands.

Vonk, R., 1993, "A Micromechanical Investigation of Softening of Concrete Loaded in Compression," *HERON*, V. 38.

Vos, E., 1983, "Influence of Loading Rate and Radial Pressure on Bond in Reinforced Concrete," PhD Thesis, Delft University Press, the Netherlands.

Wawrzynek, P. A., and Ingraffea, A. R., 1986, "Local Remeshing around a Propagating Crack Tip Using Interactive Computer Graphics," *Finite Element Method, Modeling, and New Applications*, E. M. Patten et al., eds., ASME CED-1, PVP-101, pp. 33-38.

Wawrzynek, P. A., and Ingraffea, A. R., 1987, "Interactive Finite Element Analysis of Fracture Processes: An Integrated Approach," *Theo. and Appl. Fract. Mech.*, V. 8, pp. 137-150.

Wawrzynek, P. A., and Ingraffea, A. R., 1989, "An Interactive Approach to Local Remeshing Around a Propagating Crack," *Finite Elements in Analysis and Design*, V. 5, pp. 87-96.

Westergaard, H. M., 1934, "Stresses at a Crack, Size of the Crack, and the Bending of Reinforced Concrete," ACI JOURNAL, *Proceedings* V. 30, pp. 93-102.

Wilson, W. K., 1969, "On Combined Mode Fracture Mechanics," PhD Dissertation, University of Pittsburgh.

Wu, Z.; Yoshikawa, H.; and Tanabe, T.-A., 1991, "Tension Stiffness Model for Cracked Reinforced Concrete," *J. Struct. Engrg.*, ASCE, V. 117, No. 3, Mar., pp. 715-732.

Xie, M.; Gerstle, W.; and Rahulkumar, P., 1995, "Energy-Based Automatic Mixed-Mode Crack Propagation Modeling," *J. Engrg. Mechanics*, ASCE, V. 121, No. 8, Aug., pp. 914-923.

Yamaguchi, E., and Chen, W.-F., 1990, "Cracking Model for Finite Element Analysis of Concrete Materials," *J. Engrg. Mechanics*, ASCE, V. 116, No. 6, June, pp. 1242-1261.

Yamaguchi, E., and Chen, W.-F., 1991, "Microcrack Propagation Study of Concrete under Compression," *J. Engrg. Mechanics*, ASCE, V. 117, No. 3, Mar., pp. 653-673.

Zienkiewicz, O. C., and Cheung, Y. K., 1964, "Buttress Dams on Complex Rock Foundations," *Water Power*, V. 16, p. 193.

APPENDIX—GLOSSARY

Bond-slip—The force-displacement relationship describing the mechanical interaction between a reinforcing bar and the surrounding concrete. The bond-slip relationship is a structural rather than materials phenomenon.

Continuum mechanics—A view of mechanical behavior that assumes that all responses can be represented using continuous functions.

Continuum damage mechanics (CDM)—A form of continuum mechanics that assumes that all damage is in the form of continuously distributed damage (microcracks).

Cosserat continuum—A particular view of continuum mechanics that includes the assumption of couple stresses as a continuum approximation of the behavior of crystal lattices on a small scale (Cosserat and Cosserat 1909).

Crack—A conceptual model of damage within a solid body. A crack is usually understood as a surface within the body, across which tensile tractions cannot be transmitted.

Crack band model—A method for objectively modeling cracks using the finite element method, in which the constitutive models for continuum finite elements are adjusted to correctly account for the formation of a discrete crack within the element (Bazant 1984).

Critical stress intensity factor (K_{Ic})—In LEFM, for a given material, the mode I stress-intensity factor at which a crack will propagate (Broek 1986). The critical stress-intensity factor is a material property.

Damage—Breaking of bonds within a material. Damage can be manifested as continuum damage (distributed microcracks), or as a single macrocrack. Other forms of damage are also conceivable.

Deformation—Relative movement of particles within a body, due to a variety of causes.

Discrete cracking—Damage of a deformable body which results in one or more distinct cracks. Discrete cracking models usually assume no distributed damage in the form of microcracks.

Dynamic relaxation—A numerical solution method, particularly suited to nonlinear statical problems, in which a static problem is viewed as a damped dynamic problem which converges to a steady-state static solution. The damping coefficients are adaptively adjusted to achieve rapid convergence (Papadrakis 1981).

Energy release rate (G)—The rate at which potential energy is released with respect to change in area of a discrete crack (Broek 1986). The rate of energy release may also be considered with respect to other variables.

Fictitious crack model (FCM)—A view of fracture mechanics that assumes that a discrete crack forms through gradual breaking of bonds across its advancing path (Hillerborg et al. 1976).

Finite element method (FEM)—A numerical technique for solving boundary value problems (such as structural analyses) by representing a continuum (a structure or structural member) by a group of discrete regions (finite elements) for which the behavior is described by an assumed set of functions (which, in structural analysis, relate displacements and forces).

Fracture—A fracture is a synonym with “crack.” Each word may be used as a noun or a verb.

Fracture mechanics—The study of the formation of cracks (Broek 1986). Recently, fracture mechanics also includes concepts of continuum damage mechanics.

Fracture-sensitive—A structure is fracture-sensitive if its global mechanical response is sensitive to the presence of cracks.

Fracture toughness (G_{Ic})—In LEFM, for a given material, the Mode I energy release rate, above which a crack will propagate (Broek 1986). Fracture toughness is a material property. Sometimes confused with critical stress-intensity factor.

Fracture process zone (FPZ)—The zone at the tip of a discrete crack within which damage (or some contexts, any nonlinear material behavior) occurs.

Hybrid finite element—A finite element that includes both stresses and displacements or strains as primary responses (Tong et al. 1973).

Interface—An idealized surface representing the mechanical interaction between two adjacent bodies.

Interface element—A finite element designed to model an interface (Goodman et al. 1968).

Linear elastic fracture mechanics (LEFM)—A classical branch of fracture mechanics, in which material behavior is assumed to be linear elastic, damage is considered to be only in the form of discrete cracks, and the fracture process zone is considered to be negligibly small (Broek 1986).

J-Integral—An integral method for determining energy release rates of discrete cracks (Rice 1968).

Least dimension—A parameter characterizing the geometrical size associated with a crack tip (Gerstle and Abdalla 1990).

Localization limiter—A mathematical device used to avoid unobjectivity or spurious mesh sensitivity when modeling continuum damage mechanics (Bazant and Pijaudier-Cabot 1988).

Mode—Type of loading in which stresses at a crack tip are (1) normal to the crack surface, giving rise to opening, or mode I, crack deformation perpendicular to the plane of the crack; (2) shear stresses in the crack plane, giving rise to sliding, or mode II, crack deformation in the plane of the crack, perpendicular to the edge of the crack; or (3) shear stresses out of the crack plane, giving rise to tearing, or mode III, crack deformation in the plane of the crack, parallel to the edge of the crack (Broek 1986).

Mixed-mode—Type of loading, often resulting in a curved crack, due to the simultaneous application of two or more loading modes. In most cases, mixed-mode cracks rapidly follow a direction that places them under mode I loading (Broek 1986).

Nonlocal continuum—A form of continuum damage mechanics in which localization limiters are employed to preserve solution objectivity with respect to mesh refinement (Bazant 1986).

Propagation—A term used to describe the evolution or growth of a discrete crack.

Size effect—A phenomenon associated with cracks in which the apparent nominal strength of a structure decreases as a function of structure size (Bazant and Oh 1983).

Smeared cracking—A synonym for “continuum damage mechanics.” Some smeared cracking models have been found to be nonobjective with respect to mesh refinement.

Singularity—A mathematical concept in which a field tends asymptotically to infinity at a point.

Singular crack tip element—A finite element with a strain singularity included in its shape function, for use in finite element modeling of LEFM problems (Barsoum 1976).

Stiffness-derivative method—A finite-difference method for obtaining the energy release rate of a discrete crack using the finite element method (Parks 1974).

Stress intensity factor (K_I, K_{II}, K_{III})—Factors that characterize the stress field a the tip of a crack using the assumptions of LEFM (Broek 1986).

Strain gradient—Spatial rate of change of strain, used in a method of limiting localization in continuum damage mechanics (Schreyer 1990).

Strain localization—A phenomenon associated with continuum damage mechanics, in which damage tends to concentrate along surfaces, resulting in the formation of cracks (Bazant 1976).

Strain-softening—A feature of some constitutive models, in which stress decreases with increasing strain. Constitutive models that include strain-softening must include the concept of nonlocal continuum to obtain objective solutions (Bazant 1976).

Traction—Force per unit area acting upon a specified surface. Traction is a vector quantity.

Virtual crack extension (VCE)—A finite-difference method for obtaining the energy release rate of a discrete crack using the finite element method (Hellen 1975).

This report was submitted to letter ballot by the committee and was approved in accordance with ACI balloting procedures.