Chapter 4

Dynamic Response of Structures

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Abstract: Basic principles of structural dynamics are presented with emphasis on applications to the earthquake resistant design of building structures. Dynamic characteristics of single degree of freedom systems are discussed along with their application to single story buildings. The response of these systems to harmonic and impulse loading is described and illustrated by application to simple structures. Consideration of the earthquake response of these systems leads to the concept of the elastic response spectrum and the development of design spectra. The use of procedures based on a single degree of freedom is extended to multiple degree of freedom systems through the use of the generalized coordinate approach. The determination of generalized dynamic properties is discussed and illustrated. A simple numerical integration procedure for determining the nonlinear dynamic response is presented. The application of matrix methods for the analysis of multiple degree of freedom systems is discussed and illustrated along with earthquake response analysis. A response spectrum procedure suitable for hand calculation is presented for elastic response analyses. The nonlinear static analysis for proportional loading and the nonlinear dynamic analysis for earthquake loading are discussed and illustrated with application to building structures. Finally, the use of the recorded response from buildings containing strong motion instrumentation for verification of analytical models is discussed.

4.1 Introduction

The main cause of damage to structures during an earthquake is their response to ground motions which are input at the base. In order to evaluate the behavior of the structure under this type of loading condition, the principles of structural dynamics must be applied to determine the stresses and deflections, which are developed in the structure. Structural engineers are familiar with the analysis of structures for static loads in which a load is applied to the structure and a single solution is obtained for the resulting displacements and member forces. When considering the analysis of structures for dynamic motions, the term dynamic simply means “time-varying”. Hence the loading and all aspects of the response vary with time. This results in possible solutions at each instant during the time interval under consideration. From an engineering standpoint, the maximum values of the structural response are usually the ones of particular interest, specially in the case of structural design.

The purpose of this chapter is to introduce the principles of structural dynamics with emphasis on earthquake response analysis. Attention will initially be focused on the response of simple structural systems, which can be represented in terms of a single degree of freedom. The concepts developed for these systems will then be extended to include generalized single-degree-of-freedom (SDOF) systems using the generalized-coordinate approach. This development in turn leads to the consideration of the response of structures having multiple degrees of freedom. Finally, concepts and techniques used in nonlinear dynamic-response analysis will be introduced.

4.2 Dynamic Equilibrium

The basic equation of static equilibrium used in the displacement method of analysis has the form,

$$p = kv \tag{4-1}$$

where p is the applied force, k is the stiffness resistance, and v is the resulting displacement. If the statically applied force is now replaced by a dynamic or time-varying force $p(t)$, the equation of static equilibrium becomes one of dynamic equilibrium and has the form

$$p(t) = m\ddot{v}(t) + c\dot{v}(t) + kv(t) \tag{4-2}$$

where a dot represents differentiation with respect to time.

A direct comparison of these two equations indicates that two significant changes, which distinguish the static problem from the dynamic problem, were made to Equation 4-1 in order to obtain Equation 4-2. First, the applied load and the resulting response are now functions of time, and hence Equation 4-2 must be satisfied at each instant of time during the time interval under consideration. For this reason it is usually referred to as an equation of motion. Secondly, the time dependence of the displacements gives rise to two additional forces which resist the applied force and have been added to the right-hand side.

The equation of motion represents an expression of Newton’s second law of motion, which states that a particle acted on by a force (torque) moves so that the time rate of change of its linear (angular) momentum is equal to the force (torque):

$$p(t) = \frac{d}{dt}(m\frac{dv}{dt}) \tag{4-3}$$

where the rate of change of the displacement with respect to time, *dv/dt,* is the velocity, and the momentum is given by the product of the mass and the velocity. Recall that the mass is equal to the weight divided by the acceleration of gravity. If the mass is constant, Equation 4-3 becomes

$$p(t) = m\frac{d}{dt}(\frac{dv}{dt}) = m\ddot{v}(t) \tag{4-4}$$

which states that the force is equal to the product of mass and acceleration. According to d'Alembert's principle, mass develops an inertia force, which is proportional to its acceleration and opposing it. Hence the first term on the right-hand side of Equation 4-2 is called the inertia force; it resists the acceleration of the mass.

Dissipative or damping forces are inferred from the observed fact that oscillations in a structure tend to diminish with time once the time-dependent applied force is removed. These forces are represented by viscous damping forces, that are proportional to the velocity with the constant proportionality referred to as the damping coefficient. The second term on the right-hand side of Equation 4-2 is called the damping force.

Inertia forces are the more significant of the two and are a primary distinction between static and dynamic analyses.

It must also be recognized that all structures are subjected to gravity loads such as self-weight (dead load) and occupancy load (live load) in addition to dynamic base motions. In an elastic system, the principle of superposition can be applied, so that the responses to static and dynamic loadings can be considered separately and then combined to obtain the total structural response. However, if the structural behavior becomes nonlinear, the response becomes load-path-dependent and the gravity loads must be considered concurrently with the dynamic base motions.

Under strong earthquake motions, the structure will most likely display nonlinear behavior, which can be caused by material nonlinearity and/or geometric nonlinearity. Material nonlinearity occurs when stresses at certain critical regions in the structure exceed the elastic limit of the material. The equation of dynamic equilibrium for this case has the general form

$$p(t) = m\ddot{v}(t) + c\dot{v}(t) + k(t)v(t) \qquad (4\text{-}5)$$

in which the stiffness or resistance k is a function of the yield condition in the structure, which in turn is a function of time. Geometric nonlinearity is caused by the gravity loads acting on the deformed position of the structure. If the lateral displacements are small, this effect, which is often referred to as P-delta, can be neglected. However, if the lateral displacements become large, this effect must be considered. In order to define the inertia forces completely, it would be necessary to consider the accelerations of every mass particle in the structure and the corresponding displacements. Such a solution would be prohibitively time-consuming. The analysis procedure can be greatly simplified if the mass of the structure can be concentrated (lumped) at a finite number of discrete points and the dynamic response of the structure can be represented in terms of this limited number of displacement components. The number of displacement components required to specify the position of the mass points is called the number of dynamic degrees of freedom. The number of degrees of freedom required to obtain an adequate solution will depend upon the complexity of the structural system. For some structures a single degree of freedom may be sufficient, whereas for others several hundred degrees of freedom may be required.

4.3 SINGLE-DEGREE-OF-FREEDOM SYSTEMS

4.3.1 Time-Dependent Force

The simplest structure that can be considered for dynamic analysis is an idealized, one-story structure in which the single degree of freedom is the lateral translation at the roof level as shown in Figure 4-1. In this idealization, three important assumptions are made. First, the mass is assumed to be concentrated (lumped) at the roof level. Second, the roof system is assumed to be rigid, and third, the axial deformation in the columns is neglected. From these assumptions it follows that all lateral resistance is in the resisting elements such as columns, walls, and diagonal braces located between the roof and the base. Application of these assumptions results in a discretized structure that can be represented as shown in either Figure 4-lb or 4-1c with a time-dependent force applied at the roof level. The total stiffness k is simply the sum of the stiffnesses of the resisting elements in the story level.

The forces acting on the mass of the structure are shown in Figure 4-1d. Summing the forces acting on the free body results in the following equation of equilibrium, which must be satisfied at each instant of time:

$$f_i + f_d + f_s = p(t) \qquad (4\text{-}6)$$

where

f_i = inertia force = $m\ddot{u}$

f_d = damping (dissipative) force= $c\dot{v}$

f_s = elastic restoring force = kv

$p(t)$ = time-dependent applied force

$\ddot{u}$ is the total acceleration of the mass, and $\dot{v}, v$ are the velocity and displacement of the mass relative to the base. Writing Equation *4*-6 in terms of the physical response parameters results in

$$m\ddot{u} + c\dot{v} + kv = p(t) \qquad (4\text{-}7)$$

It should be noted that the forces in the damping element and in the resisting elements depend upon the relative velocity and relative displacement, respectively, across the ends of these elements, whereas the inertia force depends upon the total acceleration of the mass. The total acceleration of the mass can be

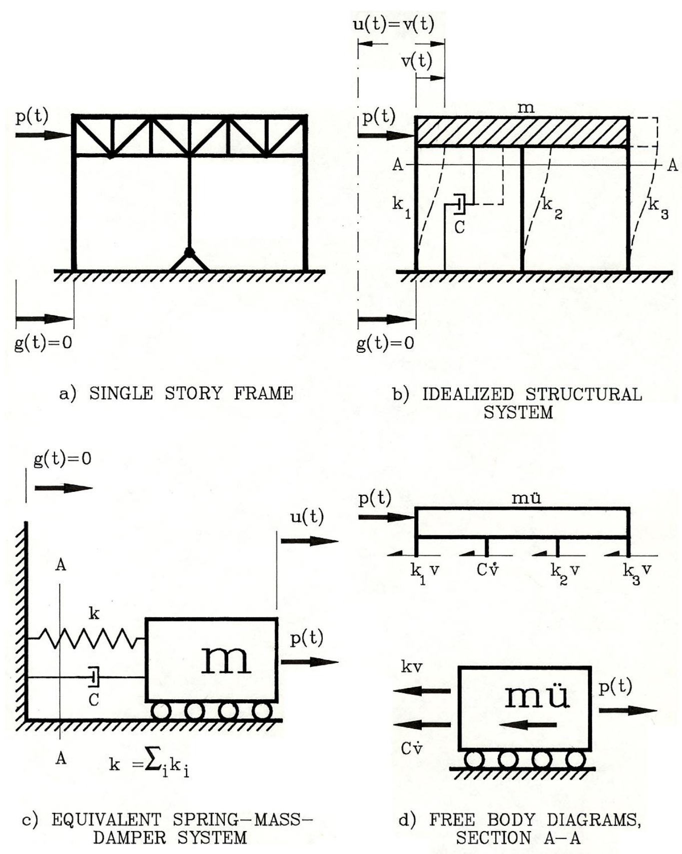

Figure 4-1. single-degree-of-freedom system subjected to time-dependent force.

expressed as

$$\ddot{u}(t) = \ddot{g}(t) + \ddot{v}(t) \tag{4-8}$$

where

$\ddot{v}(t)$ = acceleration of the mass relative to the base
$\ddot{g}(t)$ = acceleration of the base

In this case, the base is assumed to be fixed with no motion, and hence $\ddot{g}(t) = 0$ and $\ddot{u}(t) = \ddot{v}(t)$. Making this substitution for the acceleration, Equation 4-7 for a time-dependent force becomes

$$m\ddot{v} + c\dot{v} + kv = p(t) \tag{4-9}$$

4.3.2 Earthquake Ground Motion

When a single-story structure, shown in Figure 4-2a, is subjected to earthquake ground motions, no external dynamic force is applied at the roof level. Instead, the system experiences an acceleration of the base. The effect of this on the idealized structure is shown in Figure 4-2b and 4-2c. Summing the forces shown in Figure 4-2d results in the following equation of dynamic equilibrium:

$$f_i + f_d + f_s = 0 \tag{4-10}$$

Substituting the physical parameters for f_i, f_d and f_s in Equation 4-10 results in an equilibrium equation of the form

$$m\ddot{u} + c\dot{v} + kv = 0 \tag{4-11}$$

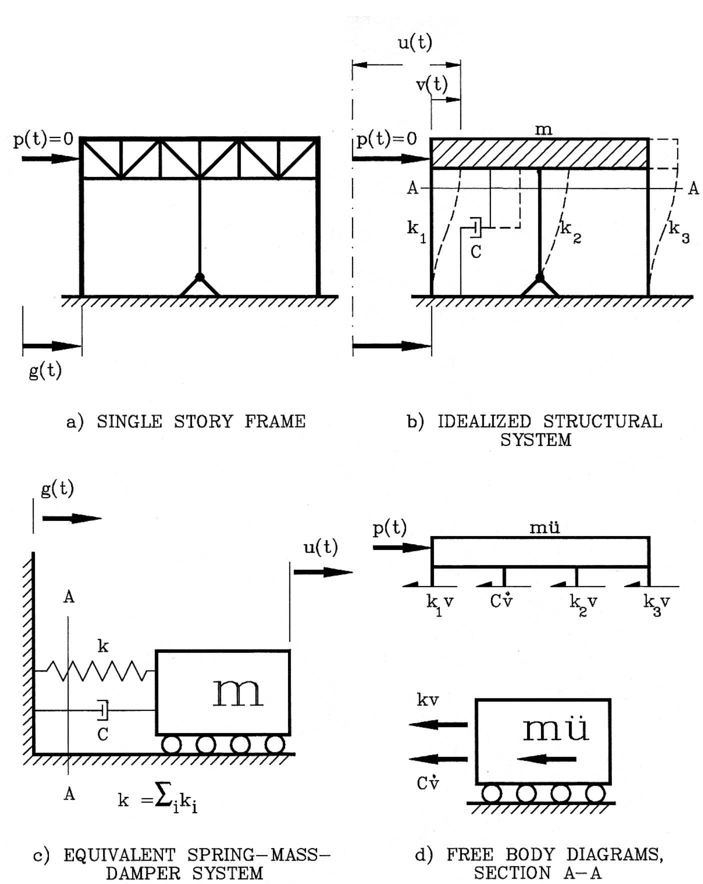

Figure 4-2. Single-degree-of-freedom system subjected to base motion.

This equation can be written in the form of Equation 4-9 by substituting Equation 4-8 into Equation 4-11 and rearranging terms to obtain

$$m\ddot{v} + c\dot{v} + kv = p_e(t) \tag{4-12}$$

where

$p_e(t)$ = effective time-dependent force
$= -m\ddot{g}(t)$

Hence the equation of motion for a structure subjected to a base motion is similar to that for a structure subjected to a time-dependent force if the base motion is represented as an effective time-dependent force which is equal to the product of the mass and the ground acceleration.

4.3.3 Mass and Stiffness Properties

Most SDOF models consider structures, which experience a transactional displacement of the roof relative to the base. In this case the translational mass is simply the concentrated weight divided by the acceleration of gravity (32.2 ft/sec^2 or 386.4 in./sec^2). However, cases do arise in which the rotational motion of the system is significant. An example of this might be the rotational motion of a roof slab which has unsymmetrical lateral supports. Newton's second law of motion states that the time rate of change of the angular momentum (moment of momentum) equals the torque. Considering a particle of mass rotating about an axis *o,* as shown in Figure 4-3, the moment of momentum can be expressed as

$$L = rm\dot{v}(t) = mr^2 \frac{d\theta}{dt} \tag{4-13}$$

The torque *N* is then obtained by taking the time derivative:

$$N = \frac{dL}{dt} = I\ddot{\theta} \tag{4-14}$$

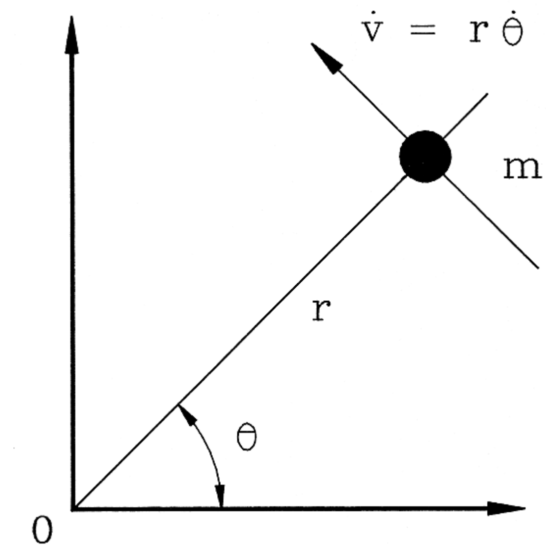

Figure 4-3. Rotating particle of mass.

where

$I = mr^2$ = mass moment of inertia

For a rigid body, the mass moment of inertia can be obtained by summing over all the mass particles making up the rigid body. This can be expressed in integral form as

$$I = \int \rho^2 dm \tag{4-15}$$

where ρ is the distance from the axis of rotation to the incremental mass *dm.* For dynamic analysis it is convenient to treat the rigid-body inertia forces as though the translational mass and the mass moment of inertia were concentrated at the center of mass. The mass and mass moment of inertia of several common rigid bodies are summarized in Figure 4-4.

Example 4-1 (Determination of Mass Properties)

Compute the mass and mass moment of inertia for the rectangular plate shown in Figure 4-5.

- Translational mass:

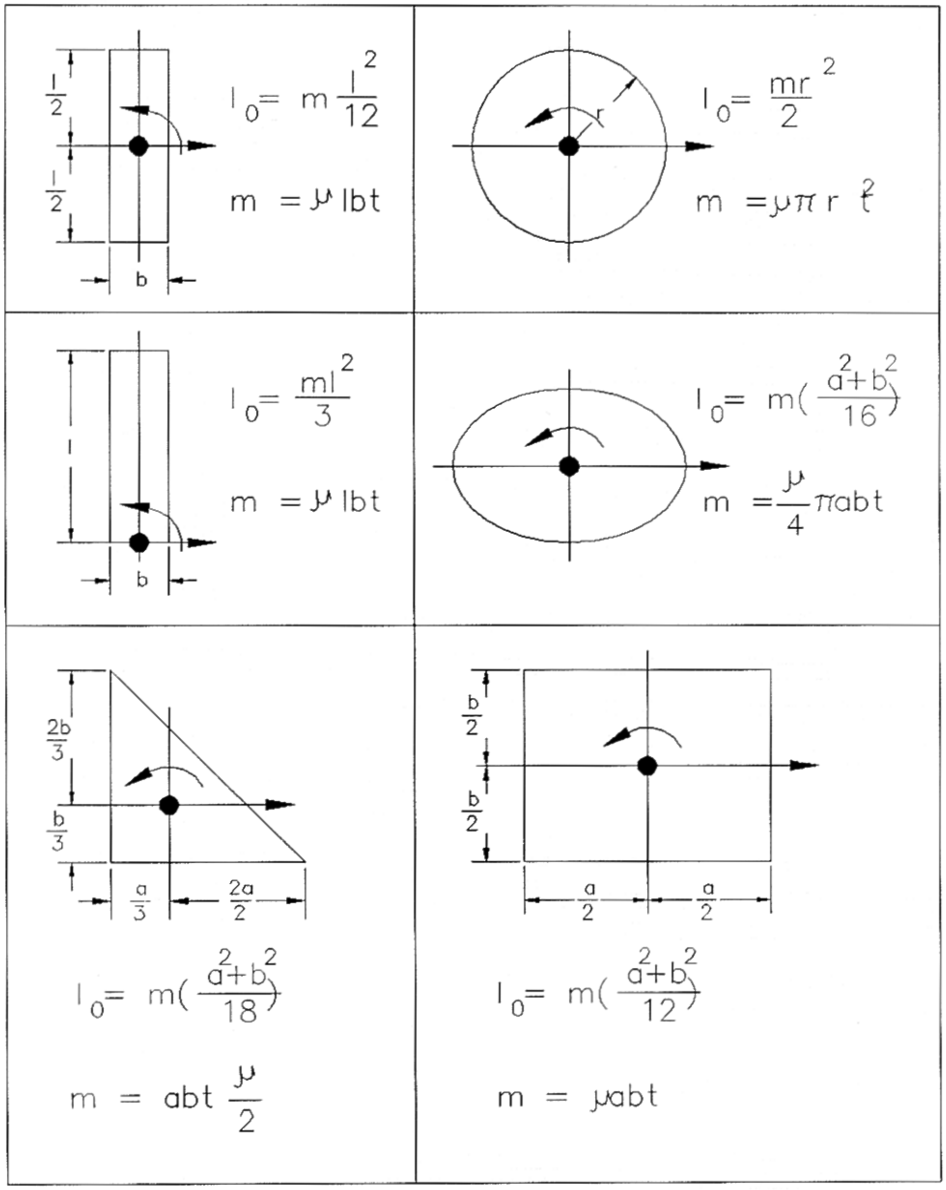

Figure 4-4. Rigid-body mass and mass moment of inertia.

$$m = \mu v = \mu abt$$

where
μ = mass density = mass per unit volume
V = total volume

• Rotational mass moment of inertia:

$$I = \int \rho^2 dm \ , \qquad \text{Where } \rho^2 = x^2 + y^2$$

$$dm = \mu dV = \mu t dx dy$$

$$I = \int \rho^2 dm = 4\mu t \int_0^{a/2} \int_0^{b/2} (x^2 + y^2) dx dy$$

$$I = 4\mu t \frac{b^3 a + a^3 b}{48} = \mu abt \frac{b^2 + a^2}{12}$$

$$I = m \frac{a^2 + b^2}{12}$$

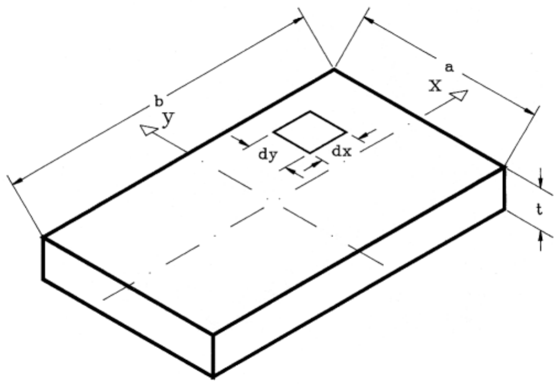

Figure 4-5. Rectangular plate of example 4-1.

In order to develop dynamic models of SDOF systems, it is necessary to review the force—displacement (stiffness) relationships of several of the more common lateral force members used in building structures. As indicated previously, the assumptions used in developing the SDOF model restrict lateral resistance to structural members between the roof and base. These might include such members as columns, diagonal braces, and walls. Stiffness properties for these elements are summarized in Figure 4-6.

4.3.4 Free Vibration

Free vibration occurs when a structure oscillates under the action of forces that are inherent in the structure without any externally

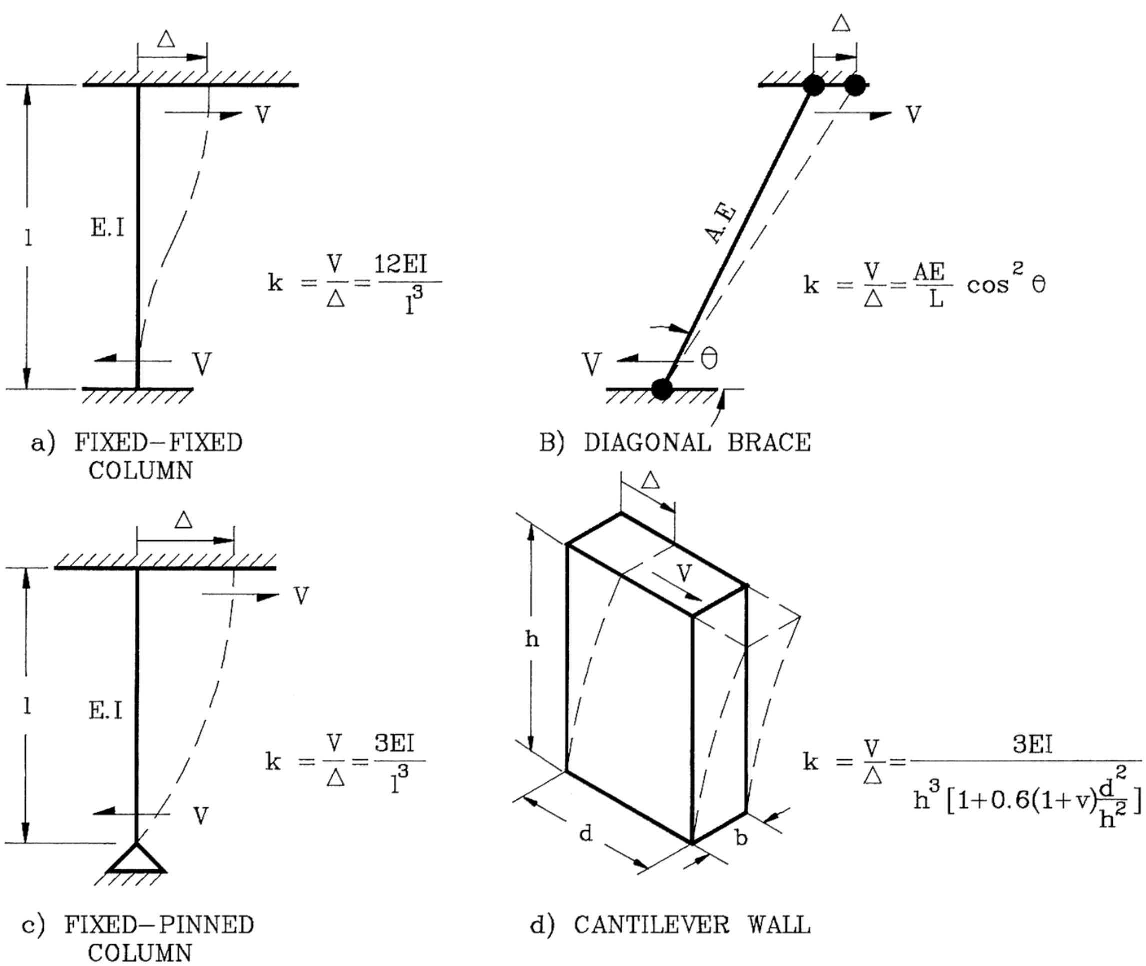

Figure 4-6. Stiffness properties of lateral force resisting elements.

applied time-dependent loads or ground motions. These inherent forces arise from the initial velocity and displacement the structure has at the beginning of the free-vibration phase.

Undamped Structures The equation of motion for an undamped SDOF system in free vibration has the form

$$m\ddot{v}(t) + kv(t) = 0 \qquad (4\text{-}16)$$

which can be written as

$$\ddot{v}(t) + \omega^2 v(t) = 0 \qquad (4\text{-}17)$$

where $\omega^2 = k/m$. This equation has the general solution

$$v(t) = A\sin\omega t + B\cos\omega t \qquad (4\text{-}18)$$

in which the constants of integration A and B depend upon the initial velocity $\dot{v}(0)$ and initial displacement $v(0)$. Applying the initial conditions, the solution has the form

$$v(t) = \frac{\dot{v}(0)}{\omega}\sin\omega t + v(0)\cos\omega t \qquad (4\text{-}19)$$

This solution in time is represented graphically in Figure 4-7.

Several important concepts of oscillatory motion can be illustrated with this result. The amplitude of vibration is constant, so that the vibration would, theoretically, continue indefinitely with time. This cannot physically be true, because free oscillations tend to diminish with time, leading to the concept of damping. The time it takes a point on the curve to make one complete cycle and return to its original position is called the period of vibration, T. The quantity ω is the circular frequency of vibration and is measured in radians per second. The cyclic frequency f is defined as the reciprocal of the period and is measured in cycles per second, or *hertz*. These three vibration properties depend only on the mass and stiffness of the structure and are related as follows:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = \frac{1}{f} \qquad (4\text{-}20a)$$

The amplitude of motion is given as:

$$p = \sqrt{\left[\frac{\dot{v}(0)}{w}\right]^2 + [v(0)]^2} \qquad (4\text{-}20b)$$

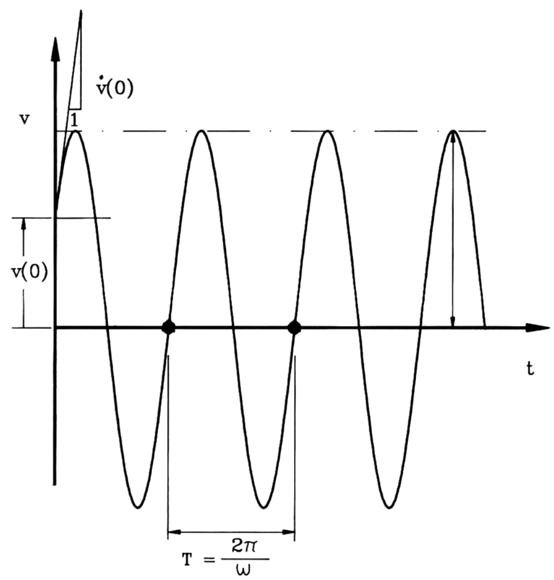

Figure 4-7. Free-vibration response of an undamped SDOF system.

It can be seen from these expressions that if two structures have the same stiffness, the one having the larger mass will have the longer period of vibration and the lower frequency. On the other hand, if two structures have the same mass, the one having the higher stiffness will have the shorter period of vibration and the higher frequency.

Example 4-2 (Period of undamped free vibration)

Construct an idealized SDOF model for the industrial building shown in Figure 4-8, and estimate the period of vibration in the two principal directions. Note that vertical cross bracings are made of 1-inch-diameter rods, horizontal cross bracing is at the bottom chord of trusses, and all columns are W8 × 24.

•Weight determination:

Roof level:

Composition roof	9.0 psf
Lights, ceiling, mechanical	6.0 psf
Trusses	2.6 psf
Roof purlins, struts	2.0 psf
Bottom chord bracing	2.1 psf
Columns (10 ft, 9 in.)	0.5 psf
Total	22.2 psf

Walls:

Framing, girts, windows	4.0 psf
Metal lath and plaster	6.0 psf
Total	10.0 psf

Total weight and mass:

$W = (22.2)(100)(75) + (10)(6)(200 + 150)$
$W = 187{,}500$ lb $= 187.5$ kips

$$m = \frac{W}{g} = \frac{187.5}{386.4} = 0.485 \text{ kips-sec}^2\text{/in.}$$

•Stiffness determination:

North—south (moment frames):

$$k_i = \frac{12EI}{L^3} = \frac{(12)(29000)(82.8)}{(144)^3}$$

$$k_i = 9.6 \text{ kips/in.}$$

$$k = \sum_{i=1}^{24} k_i = 24(9.6) = 231.6 \text{ kips/in.}$$

East—west (braced frames):

$$k_i = \frac{AE}{L}\cos^2\theta$$

$$A = \pi d^2/4 = 0.785$$

$$L = \sqrt{12^2 + 20^2} = 23.3 \text{ ft} = 280 \text{ in.}$$

$$\theta = \tan^{-1}(12/20) = 31^\circ, Cos(31^\circ) = 0.585$$

$$k_i = \frac{(0.785)(29000)(0.858)^2}{280} = 59.7 \text{ kips/in.}$$

$$k = \sum_{i=1}^{6} k_i = 6(59.7) = 358.7 \text{ kips/in.}$$

• Period determination:

North—south:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{231.6}{0.485}} = 21.8 \text{ rad/sec}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{21.8} = 0.287 \text{ sec.}$$

$$f = \frac{1}{T} = 3.48 \text{ Hz}$$

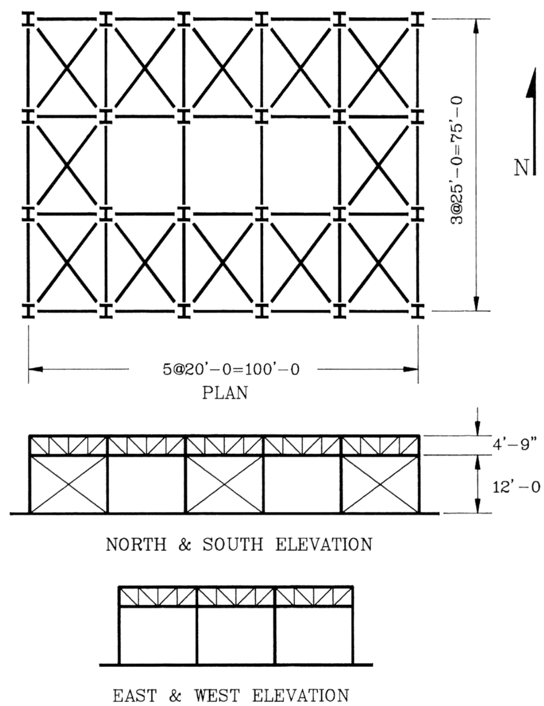

Figure 4-8. Building of Example *4*-2.

East—west:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{358.7}{0.485}} = 27.2 \text{ rad/sec}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{27.2} = 0.23 \text{ sec.}$$

$$f = \frac{1}{T} = 4.3 \text{ HZ}$$

Damped Structures In an actual structure which is in free vibration under the action of internal forces, the amplitude of the vibration tends to diminish with time and eventually the motion will cease. This decrease with time is due to the action of viscous damping forces which are proportional to the velocity. The equation of motion for this condition has the form

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = 0 \qquad (4\text{-}21)$$

This equation has the general solution

$$v(t) = e^{-\lambda\omega t}\left([\dot{v}(0) + v(0)\lambda\omega]\frac{\sin\omega_d t}{\omega_d} + v(0)\cos\omega_d t \right) \qquad (4\text{-}22a)$$

where

$\lambda = \dfrac{C}{C_{cr}} = \dfrac{C}{2m\omega}$ = percentage of critical damping

$\omega_d = \omega\sqrt{1-\lambda^2}$ = damped circular frequency

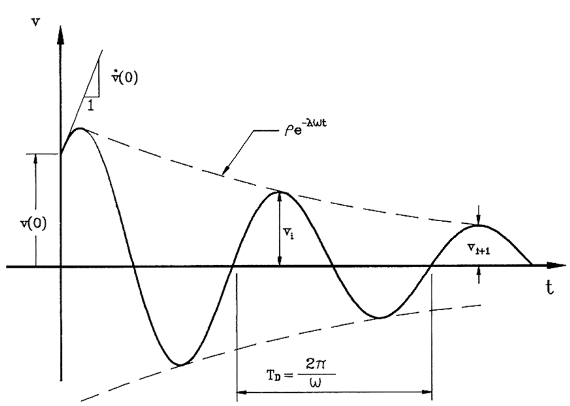

Figure 4-9. Free vibration response of a damped SDOF system.

The solution to this equation with time is shown in Figure 4-9. The damping in the oscillator is expressed in terms of a percentage of critical damping, where critical damping is defined as 2mω and is the least amount of damping that will allow a displaced oscillator to return to its original position without oscillation. For most structures, the amount of viscous damping in the system will vary between 3% and 10% of critical. Substituting an upper value of 20% into the above expression for the damped circular frequency gives the result that $\omega_d = 0.98\omega$. Since the two values are approximately the same for values of damping found in structural systems, the undamped circular frequency is used in place of the damped circular frequency. In this case the amplitude of motion is given as:

$$p = \sqrt{\left[\frac{\dot{v}(0) + v(0)\lambda w}{w_D}\right]^2 + [v(0)]^2} \qquad (4\text{-}22b)$$

One of the more useful results of the free-vibration response is the estimation of the damping characteristics of a structure. If a structure is set in motion by some external force, which is then removed, the amplitude will decay exponentially with time as shown in Figure 4-9. It can further be shown that the ratio between any two successive amplitude peaks can be approximated by the expression

$$\frac{v(i)}{v(i+1)} = e^{2\pi\lambda} \qquad (4\text{-}23)$$

Taking the natural logarithm of both sides results in

$$\delta = \ln\frac{v(i)}{v(i+1)} = 2\pi\lambda \qquad (4\text{-}24)$$

where the parameter δ is called the *logarithinic decrement.* Solving for the percentage of critical damping, λ, gives

$$\lambda \approx \frac{\delta}{2\pi} \qquad (4\text{-}25)$$

The above equation provides one of the more useful means of experimentally estimating the damping characteristics of a structure.

4.4 Response to Basic Dynamic Loading

4.4.1 Introduction

Time histories of earthquake accelerations are in general random functions of time. However, considerable insight into the response of structures can be gained by considering the response characteristics of structures to two basic dynamic loadings; harmonic loading and impulse loading. Harmonic loading idealizes the earthquake acceleration time history as a train of sinusoidal waves having a given amplitude. These might be representative of the accelerations generated by a large, distant earthquake in which the random waves generated at the source have been filtered by the soil conditions along the travel path.

Impulse loading idealizes the earthquake accelerations as a short duration impulse usually having a sinusoidal or symmetrical (isosceles) triangular shape. The idealization may be a single pulse or it may be a pulse train containing a limited number of pulses. This loading is representative of that which occurs in the near fault region.

This section will present a brief overview of the effects of harmonic loading and impulse loading on the response of building structures.

4.4.2 Harmonic Loading

For an undamped system subjected to simple harmonic loading, the equation of motion has the form

$$m\ddot{v} + kv = p_0 \sin pt \tag{3-26a}$$

where P_0 is the amplitude and p is the circular frequency of the harmonic load. For a ground acceleration, the acceleration can be represented as $\ddot{g}_0 \sin pt$, the equivalent force amplitude as $p_{oe} = m\ddot{g}_o$ and the frequency ratio $\beta = p/\omega$. The solution for the time dependent displacement has the form

$$v(t) = \frac{m\ddot{g}_o}{k} \times \frac{1}{(1-\beta^2)}(\sin pt - \beta \sin \omega t) \tag{3-26b}$$

where

$m\ddot{g}_o / k = p_{oe} / k =$ the static displacement

$\dfrac{1}{1-\beta^2}$ = dynamic amplification factor

$\sin pt$ = steady state response

$\beta \sin \omega t$ = transient response induced by the initial conditions

From equation (4-26b) it can be seen that for lightly damped systems, the peak steady state response occurs at a frequency ratio near unity when the exciting frequency of the applied load equals the natural frequency of the system. This is the condition that is called resonance. The result given in Equation (4-26b) implies that the response of the undamped system goes to infinity at resonance, however, a closer examination in the region of β equal to unity, Clough and Penzien (4-4), shows that it only tends toward infinity and that several cycles are required for the response to build up. A similar analysis for a damped system shows that at resonance, the dynamic amplification approaches a limit that is inversely proportional to the damping ratio

$$DA = \frac{1}{2\lambda} \tag{4-26c}$$

For both the undamped and the damped cases, the response builds up with the number of cycles as shown in Figure 4-10a.

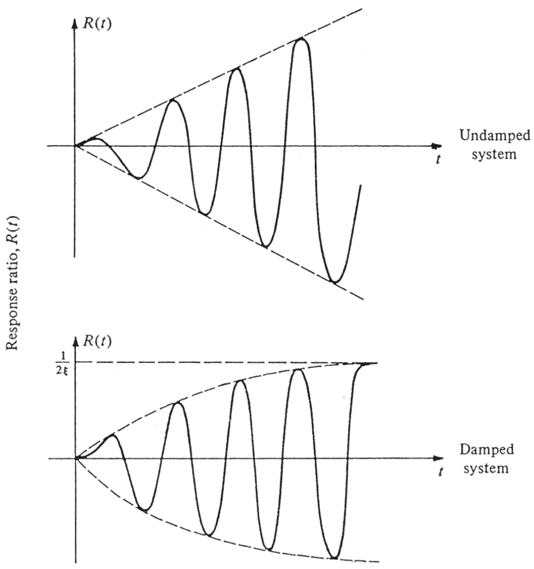

Figure 4-10a. Resonance response.

The required number of cycles for the damped case can be estimated as 1/λ. The condition of resonance can occur in buildings which are subjected to base accelerations having a frequency that is close to that of the building and having a long duration. The duration of the ground shaking is an important factor in this type of response for the reasons just discussed. The Mexico City earthquakes (1957, 1979, 1985) have produced good examples of harmonic type ground motions which have a strong resonance effect on buildings. Ground motions having a period of approximately 2 seconds were recorded during the 1985 earthquake and caused several buildings to collapse in the upper floors.

It must be recognized that as the response tends to build up, the effective damping will increase and as cracking and local yielding occur the period of the structure will shift. Both of these actions in the building will tend to reduce the maximum response. Since the dynamic amplification and number of cycles to reach the maximum response are both inversely proportional to the damping, the use of supplemental damping in the building to counter this type of ground motion is attractive.

4.4.3 Impulse Motion

Much of the initial work on impulse loads was done during the period of 1950-1965 and is discussed by Norris et al.[4-15]. The force on structures generated by a blast or explosion can be idealized as a single pulse of relatively short duration. More recently it has become recognized that some earthquake motions, particularly those in the near fault region, can be idealized as either a single pulse or as a simple pulse train consisting of one to three pulses. The accelerations recorded in Bucharest, Romania during the Vrancea, Romania earthquake (1977), shown in Figure 4-10b, are a good example of this type of motion. It is of interest to note that this site is more than 100 miles from the epicenter, indicating that this type of motion is not limited to the near fault region.

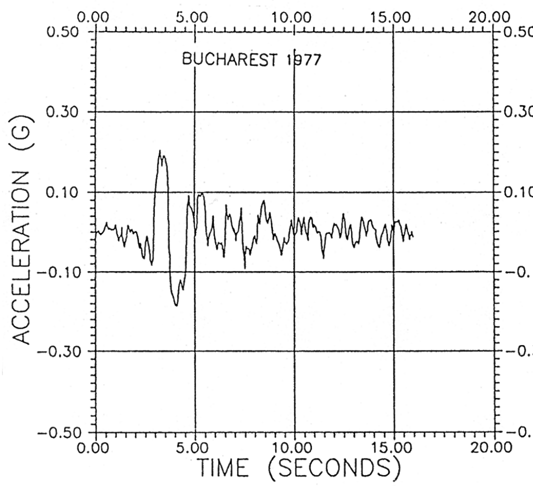

Figure 4-10b. Bucharest (1977) ground acceleration.

The maximum response to an impulse load will generally be attained on the first cycle. For this reason, the damping forces do not have time to absorb much energy from the structure. Therefore, damping has a limited effect in controlling the maximum response and is usually neglected when considering the maximum response to impulse type loads.

The rectangular pulse is a basic pulse shape. This pulse has a zero (instantaneous) rise time and a constant amplitude, p_o, which is applied to the structure for a finite duration t_d. During the time period when the load is on the structure ($t < t_d$) the equation of motion has the form

$$m\ddot{v} + kv = p_o \tag{4-26d}$$

which has the general solution

$$v(t) = \frac{p_0}{k}(1 - \cos\omega t) \tag{4-26e}$$

When the impulse load is no longer acting on the structure, the system is responding in free vibration and the equation of motion becomes

$$v(t) = \frac{\dot{v}(t_d)}{\omega}\sin\omega\bar{t} + v(t_d)\cos\omega\bar{t} \tag{4-26f}$$

where

$$\bar{t} = t - t_d$$

The displacement, $v(t_d)$ and the velocity $\dot{v}(t_d)$ at the end of the loading phase become the initial conditions for the free vibration phase. It can be shown that the dynamic amplification, DA, which is defined as the ratio of the maximum dynamic displacement to the static displacement, will equal 2 if $t_d \geq T/2$ and will equal $2\sin(\pi t_d/T)$ if $t_d \leq T/2$. For elastic response, the dynamic amplification is a function of the shape of the impulse load and the duration of the load relative to the natural period of the structure as shown in Figure 4-10c.

For nonlinear behavior, the equation of motion becomes more complex, requiring the use of numerical methods for solution. Results of initial studies for basic pulse shapes were presented in the form of response charts(4-15) such as the one shown in Figure 4-10d which can be thought of as a constant strength response spectra. For nonlinear response, the dynamic amplification factor is replaced by the displacement ductility ratio which is defined as the ratio of the maximum displacement to the displacement at yield.

$$\mu = \frac{v_{\max}}{v_{yield}} \tag{4-26g}$$

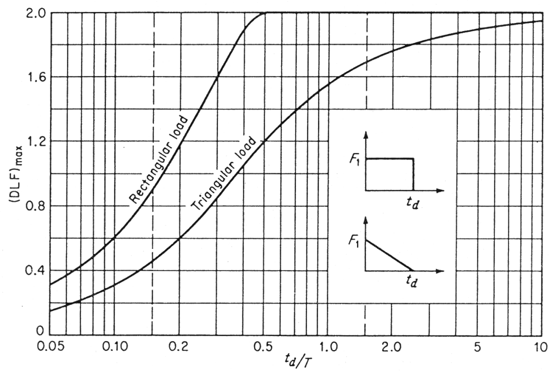

Figure 4-10c. Maximum elastic response, rectangular and triangular load pulses.[*4*-16]

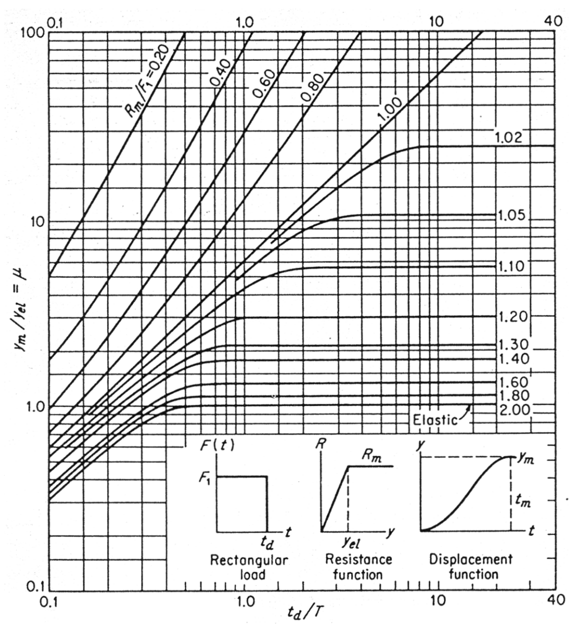

Figure 4-10d. Maximum elasto-plastic response, rectangular load pulse.[*4*-16]

It can also be seen that the single curve representing the elastic response becomes a family of curves for the inelastic response.

These curves depend upon the ratio of the maximum system resistance, R_m, to the maximum amplitude of the impulse load. Note that the bottom curve in Figure 4-10d which has a resistance ratio of 2 represents the elastic response curve with the ductility equal to or less than unity for all values of t_d/T. It can also be seen that as the resistance ratio decreases, the ductility demand increases.

4.4.4 Example 4-3 (Analysis for Impulse Base Acceleration)

The three bay frame shown in Figure 4-10e is assumed to be pinned at the base. It is subjected to a ground acceleration pulse which has an amplitude of 0.5g and a duration of 0.4 seconds. It should be noted that this acceleration pulse is similar to one recorded at the Newhall Fire Station during the Northridge earthquake (1994). The lateral resistance at ultimate load is assumed to be elasto-plastic. The columns are W10×54 with a clear height of 15 feet and the steel is A36 having a nominal yield stress of 36 ksi. Estimate the following:

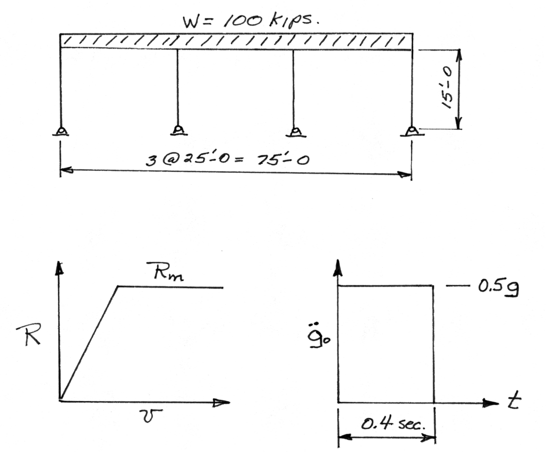

Figure 4-10e. Building elevation, resistance and loading, Example 4-3.

(a) the displacement ductility demand, (b) the maximum displacement and (c) the residual displacement.

For a W10×54 column, I = 303 in^4 and Z = 66.6 in^3 The lateral stiffness of an individual column is calculated as

$$k_i = \frac{3EI}{L^3} = \frac{3(29000)\times 303}{(15\times 12)^3} = 4.5\frac{kip}{in}$$

and the total stiffness becomes

$$K = \sum k_i = 4\times 4.5 = 18.0\frac{\text{kip}}{\text{in}}$$

The mass is the weight divided by the acceleration of gravity,

$$m = \frac{W}{g} = \frac{100\,\text{kips}}{386.4\,\frac{\text{in}}{\text{sec}^2}} = 0.26\frac{\text{kips}-\text{sec}^2}{\text{in}}$$

The period of vibration of the structure can now be calculated as

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.26}{18.0}} = 0.75\,\text{sec}.$$

and the duration ratio becomes

$$\frac{t_d}{T} = \frac{0.4}{0.75} = 0.53$$

The effective applied force, P_e is given as

$$P_e = m\ddot{g}_o = m\times 0.5g = 0.5W = 50\,\text{kips}$$

The ultimate lateral resistance of the structure occurs when plastic hinges form at the tops of the columns and a sway mechanism is formed. The nominal plastic moment capacity of a single column is

$$M_P = F_y Z = 36.\times 66.6 = 2400\,\text{in - kips}$$

and the shear resistance is

$$V_i = \frac{M_P}{h} = \frac{2400}{180} = 13.33\,\text{kips}.$$

The total lateral resistance is

$$R = 4V_i = 53.33\,Kips$$

The resistance to load ratio, is then given as

$$\frac{R}{P_{ev}} = \frac{53.3}{50} = 1.1$$

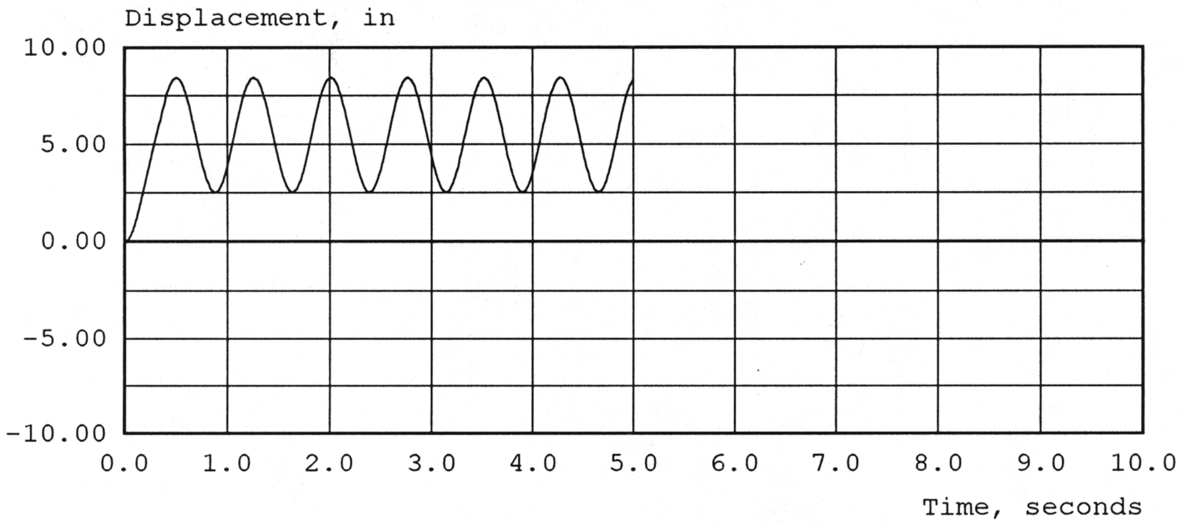

Figure 4-10f. Computed displacement time history

Using this ratio and the duration ratio, t_d/T and entering the response spectrum given in Figure 4-10d, the displacement ductility demand is found to be 2.7. The displacement at yield can be obtained as

$$v_y = \frac{R}{K} = \frac{53.3}{18} = 3.0\,in.$$

and the maximum displacement is

$$v_{\max} = \mu \times lc_y = 2.7 \times 3.0 = 8.1\text{in.}$$

The residual or plastic deformation is the difference between the maximum displacement and the displacement at yield.

$$v_{(residual)} = v_p = 8.1 - 3.0 = 5.1\,inches$$

More recently, these calculations have been programmed for interactive computation on personal computers. The program NONLIN [4-14] can be used to do this type of calculation and to gain additional insight through the graphics that are available. Using the program, the maximum displacement ductility is calculated to be 2.85, the maximum displacement is 8.4 inches, and the plastic displacement is 5.6 in. A plot of the calculated time history of the displacement, shown in Figure 4-10f, indicates that structure reaches the maximum displacement on the first cycle and that from this time onward, it oscillates about a deformed position of 5.6 inches which is the plastic displacement. This can also be seen in a plot of the force versus displacement, shown in Figure 4-10g which indicates a single yield excursion followed by elastic oscillations about the residual displacement of 5.6 inches.

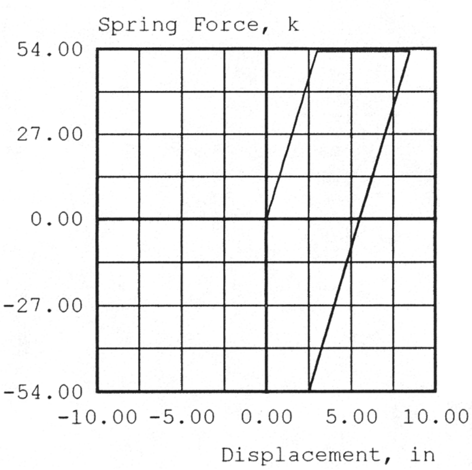

Figure 4-10g. Computed force versus displacement.

4.4.5 Approximate resopnse to impulse loading

In order to develop a method for evaluating the response of a structural system to a general dynamic loading, it is convenient to first consider the response of a structure to a short-duration impulse load as shown in Figure 4-10h, If the duration of the applied impulse load, *t*, is short relative to the fundamental period of vibration of the structure, *T*, then the effect of the impulse can be considered as an incremental change in velocity. Using the impulse-momentum relationship, which states that the impulse is equal to the change in momentum, the following equation is obtained:

$$\dot{v}(t) = \frac{1}{m}\int_0^t p(t)dt \tag{4-26}$$

Following the application of the short-duration impulse load, the system is in free vibration and the response is given by Equation 4-19. Applying the initial conditions at the beginning of the free vibration phase,

$$\dot{v}(t_1) = \frac{1}{m}\int_0^{t_1} p(t)dt, \qquad v(t_1) \text{ negligible}$$

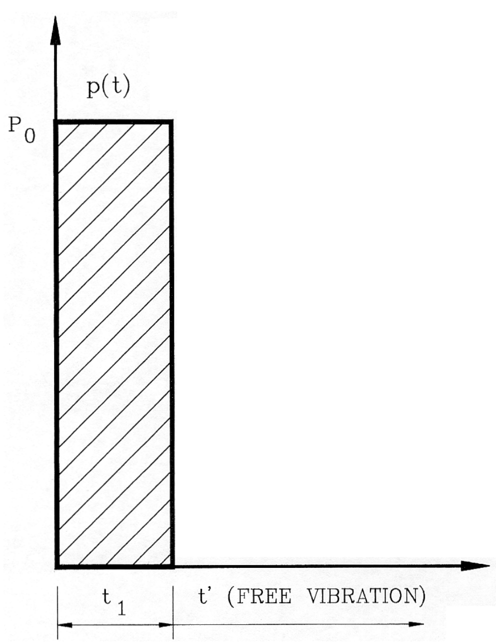

Figure 4-10h. Short duration rectangular impulse.

Equation 4-19 becomes

$$v(t-t_1) = \frac{1}{m\omega}\int_0^{t_1} p(t)\,dt \sin\omega(t-t_1) \tag{4-27}$$

For a damped structural system, the free-vibration response is given by Equation 4-22 Applying the above initial conditions to Equation 4-22 results in the following equation for the damped response:

$$v(t-t_1) = \frac{1}{m\,\omega_d}\int_0^{t_1} p(t)\,dt\, e^{-\lambda\omega(t-t_1)} \times \sin\omega_d(t-t_1) \tag{4-28}$$

4.4.6 Response to General Dynamic Loading

The above discussion of the dynamic response to a short-duration impulse load can readily be expanded to produce an analysis procedure for systems subjected to an arbitrary loading time history. Any arbitrary time history can be represented by a series of short-duration impulses as shown in Figure 4-11. Consider one of these impulses which begins at time $\Im$ after the beginning of the time history and has a duration $d\tau$. The magnitude of this differential impulse is $p(\tau)\,d\tau$, and it produces a differential response which is given as

$$dv(\tau) = \frac{p(\tau)\sin\omega t' d\tau}{m\omega} \tag{4-29}$$

The time variable t' represents the free-vibration phase following the differential impulse loading and can be expressed as

$$t' = t - \tau \tag{4-30}$$

Substituting this expression into Equation 4-29 results in

$$dv(\tau) = \frac{p(\tau)\sin\omega(t-\tau)\,d\tau}{m\omega} \tag{4-31}$$

The total response can now be obtained by superimposing the incremental responses of all the differential impulses making up the time history. Integrating Equation 4-31, the total displacement response becomes

$$v(t) = \frac{1}{m\omega}\int_0^t p(\tau)\sin\omega(t-\tau)\,d\tau \qquad (4\text{-}32)$$

which is known as the Duhamel integral. When considering a damped structural system, the differential response is given by Equation 4-28 and the Duhamel integral solution becomes

$$v(t) = \int_o^t \frac{p(\tau)\,e^{-\lambda\omega(t-\tau)}\sin\omega_d(t-\tau)\,d\tau}{m\omega_d} \qquad (4\text{-}33)$$

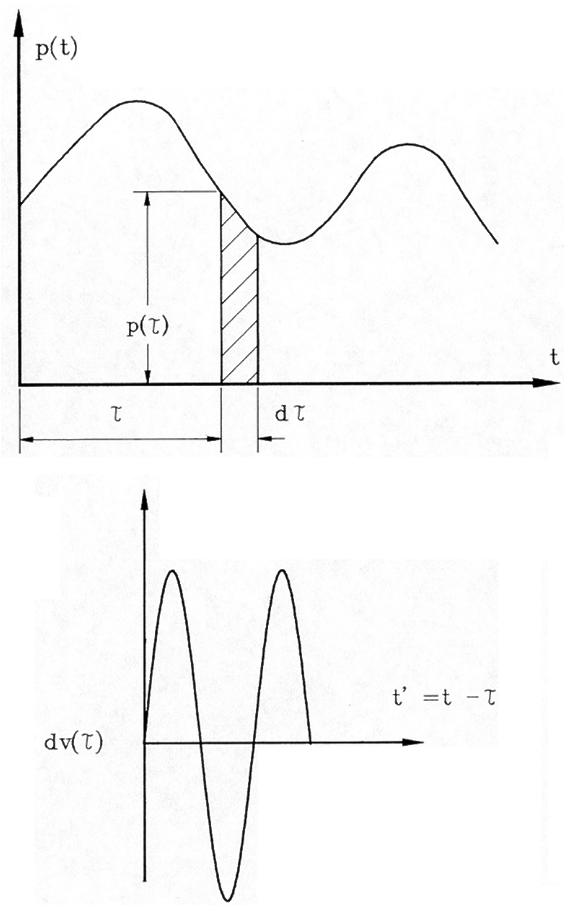

Figure 4-11. Differential impulse response.

Since the principle of superposition was used in the derivation of Equations 4-32 and 4-33, the results are only applicable to linear structural systems. Furthermore, evaluation of the integral will require the use of numerical methods. For these two reasons, the use of a direct numerical integration procedure may be preferable for solving for the response of a dynamic system subjected to general dynamic load. This will be addressed in a later section on nonlinear response analysis. However, the Duhamel-integral result can be applied in a convenient and systematic manner to obtain a solution for the linear elastic structural response for earthquake load.

4.4.7 Earthquake Response of Elastic Structures

Time-History Response The response to earthquake loading can be obtained directly from the Duhamel integral if the time-dependent force *p(t)* is replaced with the effective time-dependent force $P_e(t)$, which is the product of the mass and the ground acceleration. Making this substitution in Equation 4-33 results in the following expression for the displacement:

$$v(t) = \frac{V(t)}{\omega} \qquad (4\text{-}34)$$

where the response parameter *V(t)* represents the velocity and is defined as

$$V(t) = \int_0^t \ddot{g}(\tau)\,e^{-\lambda\,\omega(t-\tau)}\sin\omega_d(t-\tau)\,d\tau \quad (4\text{-}35)$$

The displacement of the structure at any instant of time during the entire time history of the earthquake under consideration can now be obtained using Equation 4-34. It is convenient to express the forces developed in the structure during the earthquake in terms of the effective inertia forces. The inertia force is the product of the mass and the total acceleration. Using Equation 4-11, the total acceleration can be expressed as

$$\ddot{u}(t) = -\frac{c}{m}\dot{v}(t) - \frac{k}{m}v(t) \qquad (4\text{-}36)$$

If the damping term can be neglected as contributing little to the equilibrium equation, the total acceleration can be approximated as

$$\ddot{u}(t) = -\omega^2 v(t) \tag{4-37}$$

The effective earthquake force is then given as

$$Q(t) = m\omega^2 v(t) \tag{4-38}$$

The above expression gives the value of the base shear in a single-story structure at every instant of time during the earthquake time history under consideration. The overturning moment acting on the base of the structure can be determined by multiplying the inertia force by the story height:

$$M(t) = hm\omega^2 v(t) \tag{4-39}$$

Response Spectra Consideration of the displacements and forces at every instant of time during an earthquake time history can require considerable computational effort, even for simple structural systems. As mentioned previously, for many practical problems and especially for structural design, only the maximum response quantities are required. The maximum value of the displacement, as determined by Equation 4-34, will be defined as the spectral displacement

$$S_d = v(t)_{\max} \tag{4-40}$$

Substituting this result into Equations 4-38 and 4-39 results in the following expressions for the maximum base shear and maximum overturning moment in a SDOF system:

$$Q_{\max} = m\omega^2 S_d \tag{4-41}$$

$$M_{\max} = hm\omega^2 S_d \tag{4-42}$$

An examination of Equation 4-34 indicates that the maximum velocity response can be approximated by multiplying the spectral displacement by the circular frequency. This response parameter is defined as the *spectral pseudovelocity* and is expressed as

$$S_{pv} = \omega S_d \tag{4-43}$$

In a similar manner, Equation 4-37 indicates that the maximum total acceleration can be approximated as the spectral displacement multiplied by the square of the circular frequency. This product is defined as the *spectral pseudoacceleration* and is expressed as

$$S_{pa} = \omega^2 S_d \tag{4-44}$$

A plot of the spectral response parameter against frequency or period constitutes the response spectrum for that parameter. A schematic representation of the computation of the displacement spectrum for the north-south component of the motion recorded at El Centro on May 18, 1940 has been presented by Chopra[(4-1)] and is shown in Figure 4-12. Because the three response quantities are related to the circular frequency, it is convenient to plot them on a single graph with log scales on each axis. This special type of plot is called a tripartite log plot. The three response parameters for the El Centro motion are shown plotted in this manner in Figure 4-13. For a SDOF system having a given frequency (period) and given damping, the three spectral response parameters for this earthquake can be read directly from the graph.

Two types of tripartite log paper are used for plotting response spectra. Note that on the horizontal axis at the bottom of the graph in Figure 4-13, the period is increasing from left to right. For this reason, this type of tripartite log paper is often referred to as *period* paper. A similar plot of the response spectra for the El Centro N-S ground motion is shown in Figure 4-14. Here it can be seen that frequency, plotted on the horizontal axis, is increasing from left to right. This type of tripartite paper is referred to as *frequency* paper.

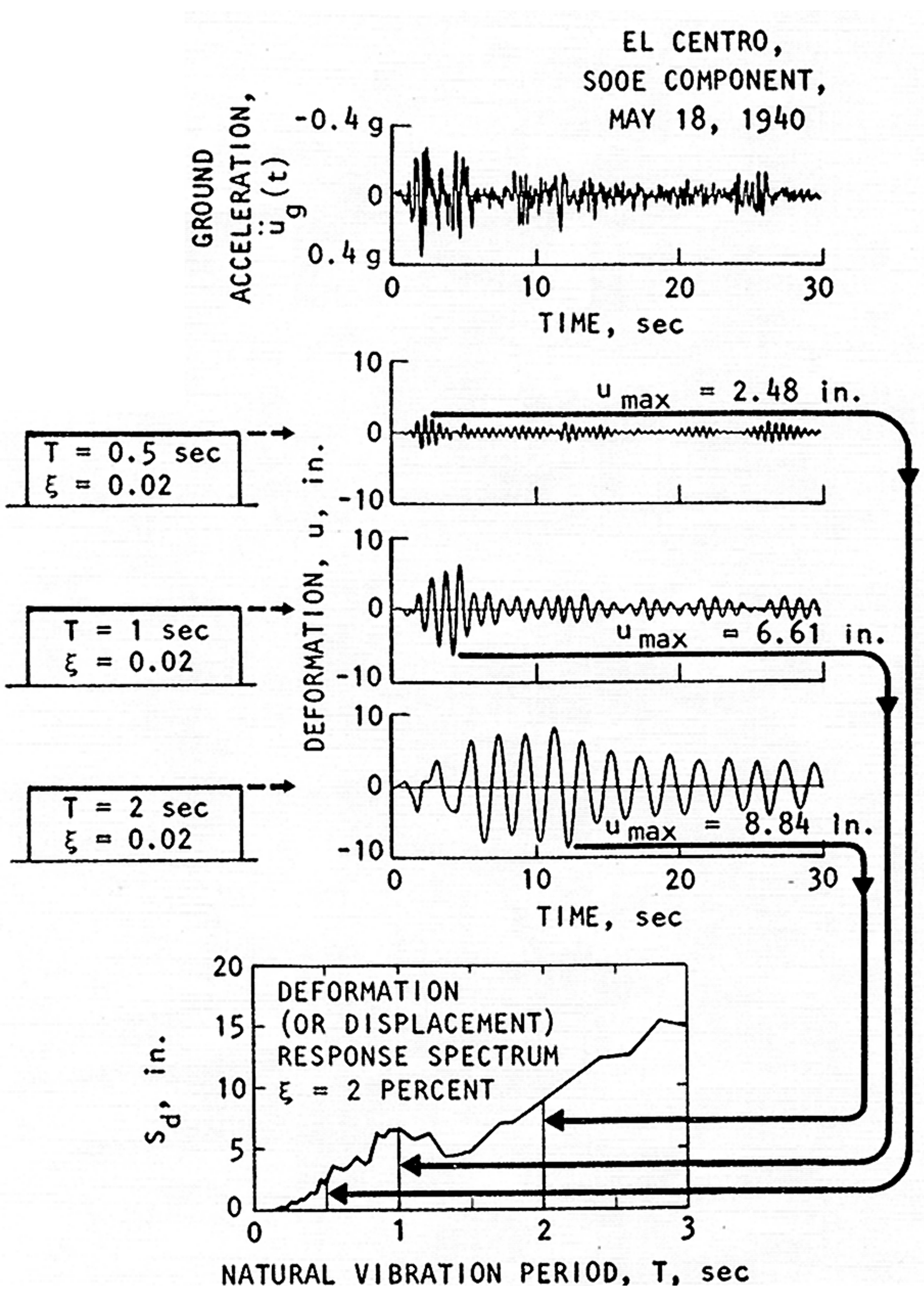

Figure 4-12. Computation of deformation (or displacement) response spectrum. [After Chopra (4-1)].

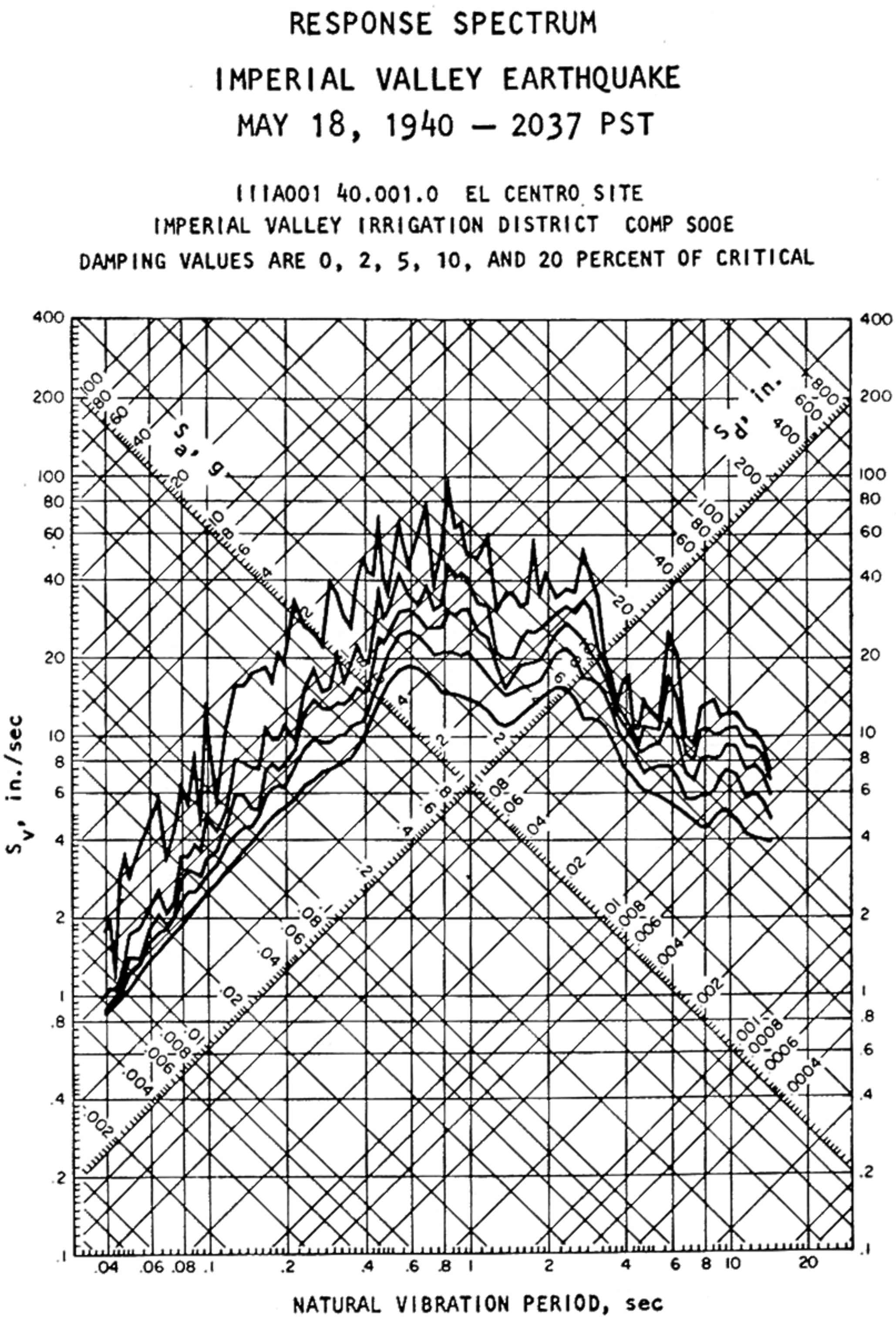

Figure 4-13. Typical tripartite response-spectra curves.

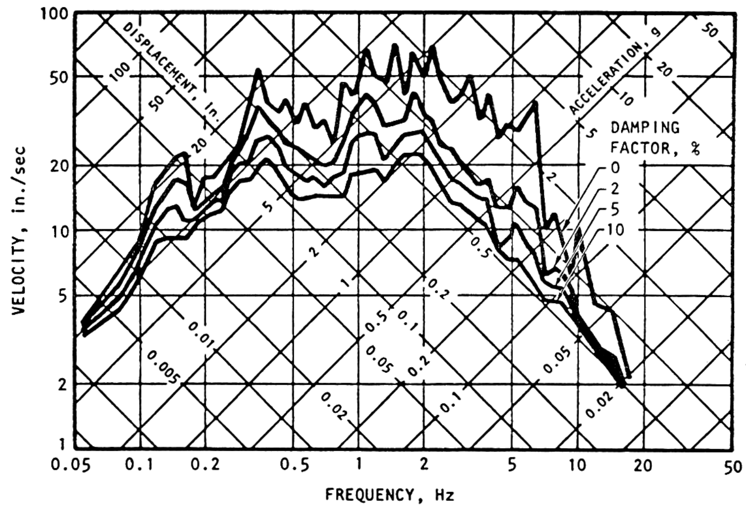

Figure 4-14. Response spectra, El Centro earthquake, May 18,1940, north-south direction.

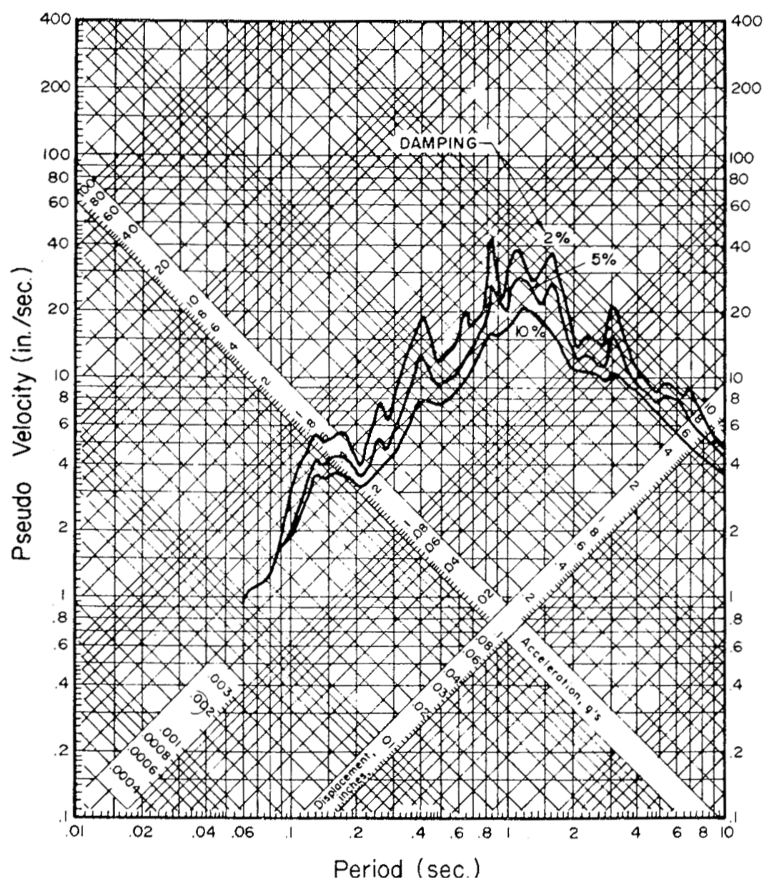

Figure 4-15. Site-specific response spectra.

4.4.8 Design Response Spectra

Use of the elastic response spectra for a single component of a single earthquake record (Figure 4-13), while suitable for purposes of analysis, is not suitable for purposes of design. The design response spectra for a particular site should not be developed from a single

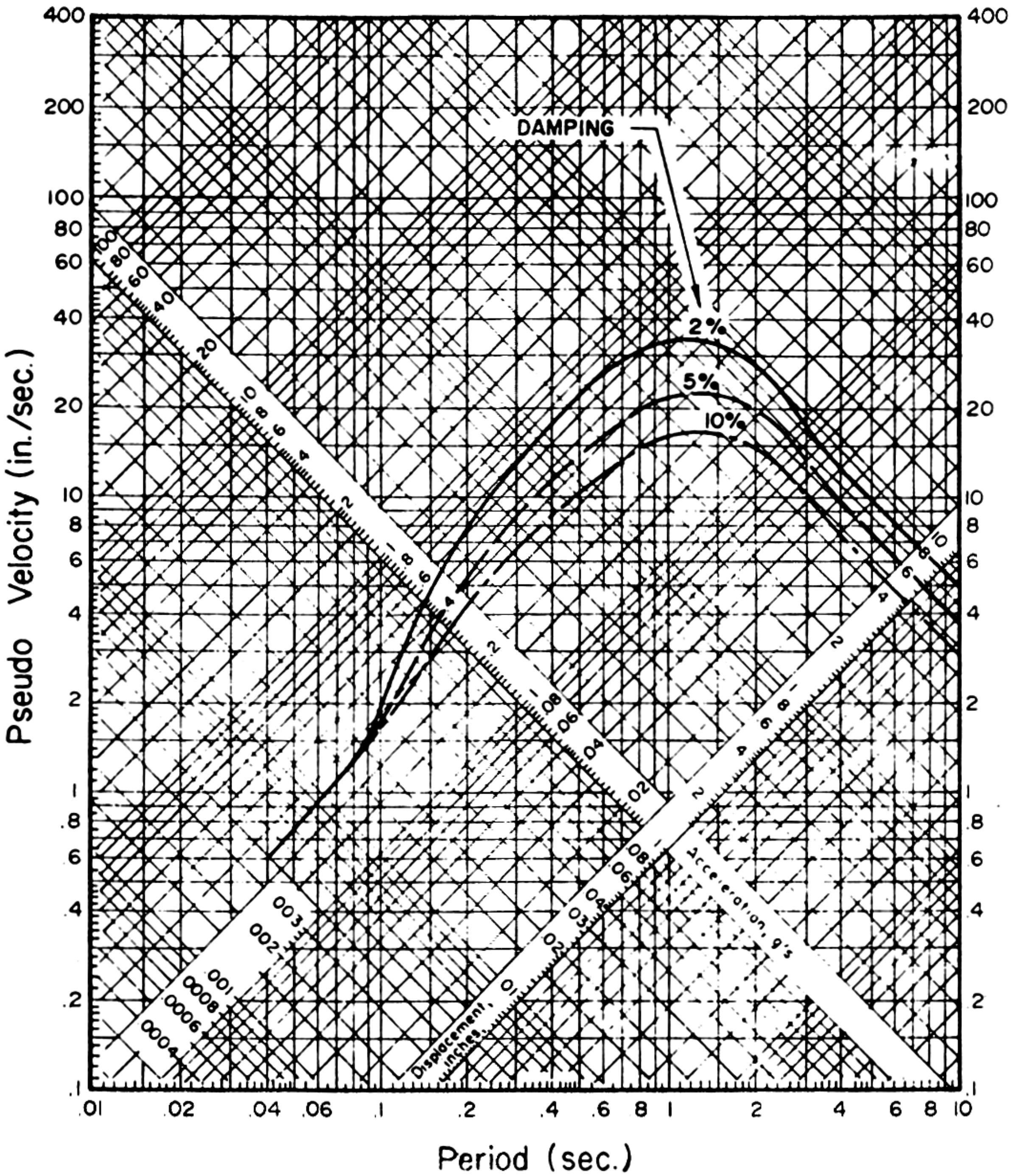

Figure 4-16. Smoothed site-specific design spectra.

acceleration time history, but rather should be obtained from the ensemble of possible earthquake motions that could be experienced at the site. This should include the effect of both near and distant earthquakes. Furthermore, a single earthquake record has a particular frequency content which gives rise to the jagged, sawtooth appearance of peaks and valleys shown in Figure 4-13. This feature is also not suitable for design, since for a given period, the structure may fall in a valley of the response spectrum and hence be underdesigned for an earthquake with slightly different response characteristics. Conversely, for a small change in period, the structure might fall on a peak and be overdesigned. To alleviate this problem the concept of the smoothed response spectrum has been introduced for design. Statistics are used to create a smoothed spectrum at some suitable design level. The mean value or median spectrum can generally be used for earthquake-resistant design of normal building structures. Use of this spectrum implies there is a 50% probability that the design level will be exceeded.

Structures that are particularly sensitive to earthquakes or that have a high risk may be designed to a higher level such as the mean plus one standard deviation, which implies that the probability of exceedance is only 15.9%. Structures having a very high risk are often designed for an enveloping spectrum which envelopes the spectra of the entire ensemble of possible site motions. Response spectra which are representative of a magnitude-6.5 earthquake at a distance of 15 miles, developed by the Applied Technology Council [4-2], are shown in Figure 4-15. The corresponding smoothed design spectra are shown in Figure 4-16.

Newmark and Hall [4-3] have proposed a method for constructing an elastic design response spectrum in which the primary input datum is the anticipated maximum ground acceleration. The corresponding values for the maximum ground velocity and the maximum ground displacement are proportioned relative to the maximum ground acceleration, which is normalized to 1.0g. The maximum ground velocity is taken as 48 in./sec, and the maximum ground displacement is taken as 36 in. It should be noted that these values represent motions which are more intense than those normally considered for earthquake-resistant design; however, they are approximately in the correct proportion for earthquakes occurring on competent soils and can be scaled for earthquakes having lower ground acceleration.

Table 4-1. Relative values of spectrum amplification factors (*4*-3).

Percentage of critical Damping	Amplification factor for		
	Displacement	Velocity	Acceleration
0	2.5	4.0	6.4
0.5	2.2	3.6	5.8
1	2.0	3.2	5.2
2	1.8	2.8	4.3
5	1.4	1.9	2.6
10	1.1	1.3	1.5
20	1.0	1.1	1.2

Three principal regions of the response spectrum are identified, in which the structural response can be approximated as a constant, amplified value. Amplification factors are applied to the ground motions in these three regions to obtain the design spectrum for a SDOF elastic system. Based on a large data base of recorded earthquake motions, amplification factors which give a probability of exceedance of about 10% or less are given in Table 4-1 for various values of the structural damping. The basic shape of the Newmark—Hall design spectrum using the normalized ground motions and the amplification factors given in Table 4-1 for 5% damping is shown in Figure 4-17. The displacement region is the low-frequency region with frequencies less than 0.33 Hz (periods greater than 3.0 sec). The maximum displacement of the SDOF system is obtained by multiplying the maximum ground displacement by the displacement amplification factor given in Table 4-1. The velocity region is in the mid-frequency region between 0.33 Hz (3.0 sec) and 2.0 Hz (0.5 sec). Maximum velocities in this region are obtained by multiplying the maximum ground velocity by the amplification factor for the velocity (Table 4-1). An amplified acceleration region lies between 2.0 Hz (0.5 sec) and 6.0 Hz (0.17 sec). The amplified response is obtained in the same manner as in the previous two cases. Structures having a frequency greater than 30 Hz (period less than 0.033 sec) are considered to be rigid and have an acceleration which is equal to the ground acceleration. In the frequency range between 6 Hz (0.17 sec) and 30 Hz (0.033 sec) there is a transition region between the ground acceleration and the amplified acceleration region.

Similar design spectra corresponding to the postulated ground motion presented in Figures 4-15 and 4-16 are shown in Figure 4-18. In order to further define which response spectrum should be used for design, it is necessary to estimate the percentage of critical damping in the structure. A summary of recommended damping values for different types of structures and different stress conditions is given in Table 4-2 as a guideline.

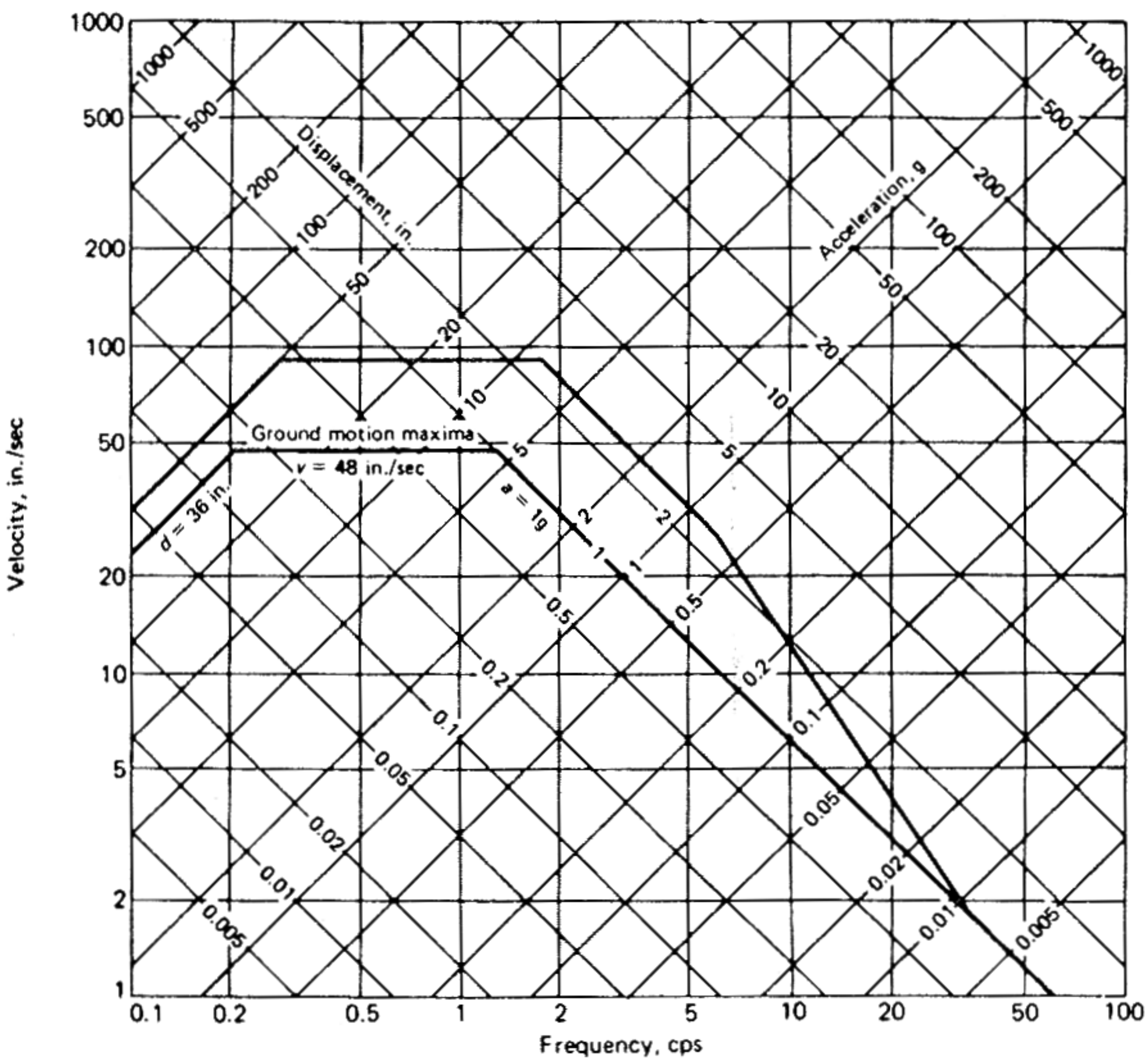

Figure 4-17. Basic New mark-Hall design spectrum normalized to 1.0g for 5% damping (4-3).

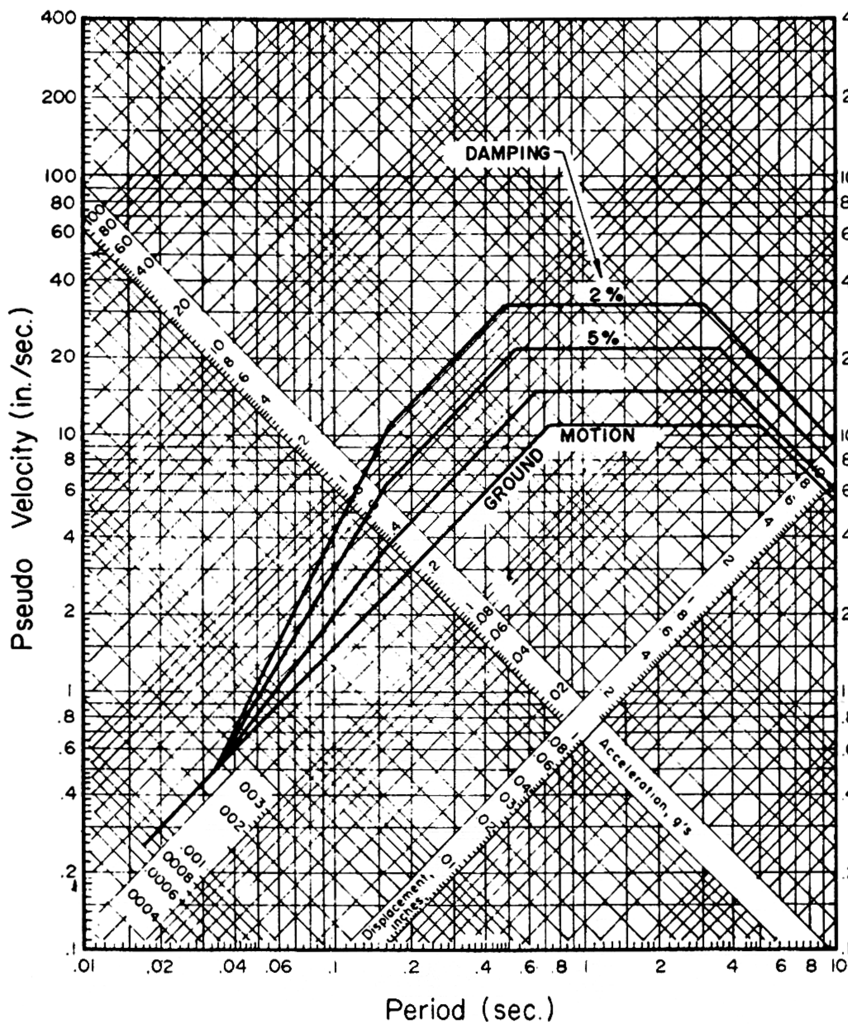

Figure 4-18. A New mark-Hall design spectra.

Example 4-4 (Construction of a Newmark-Hall Design Spectrum)

Construct a Newmark-Hall design spectrum for a maximum ground acceleration of 0.2g, and use it to estimate the maximum base shear for the industrial building of Example 4-1. Assume the damping is 5 percent of critical.

- Determine ground motion parameters:
  ground acceleration = (1.0)(0.2) = 0.2g
  ground velocity = (48.0)(0.2)=9.6in./sec.
  ground displacement=(36.0)(0.2)=7.2 in.

- Amplified response parameters:
  acceleration = (0.2)(2.6) = 0.52g
  velocity = (9.6)(1.9) = 18.2 in./sec
  displacement = (7.2)(1.4) = 10.0 in.

The constructed design spectrum is shown in Figure 4-19.

From Example 4-1:
N-S:
T = 0.287 sec.
ω = 21.8 rad/sec,
f = 3.48 HZ

From the design spectrum for f = 3.48 Hz:
$S_d = v(t)_{max}$ =0.42 in.

Table 4-2 Recommended Damping Values (4-3)

Stress level	Type and condition of structure	Percentage of critical damping	Stress level	Type and condition of structure	Percentage of critical damping
Working stress,<1/2 yield point	Vital piping	1-2	At or just below yield point	Vital piping	2-3
	Welded steel, prestressed concrete, well-reinforced concrete(only slight cracking)	2-3		Welded steel, prestressed concrete(without complete loss in prestress)	5-7
	Reinforced concrete with considerable cracking	3-5		Prestressed concrete with no prestress left	7-10
	Bolted and / or riveted steel, wood structures with nailed or bolted joints.	5-7		Bolted and / or riveted steel, wood structures with nailed or bolted joints.	10-15
				Wood structures with nailed joints	15-20

From Equation 4-42:

Q_{max} = (0.485)(21.8)^2(0.42) = 96.8 kips
E-W:
T = 0.23 sec,
ω = 27.2 rad/sec,
From the design spectrum for f = 4.3 Hz:

S_d = 0.28 in.

From Equation 4-42:

Q_{max}=(0.485)(21.8)^2(0.28) = 64.5 kips

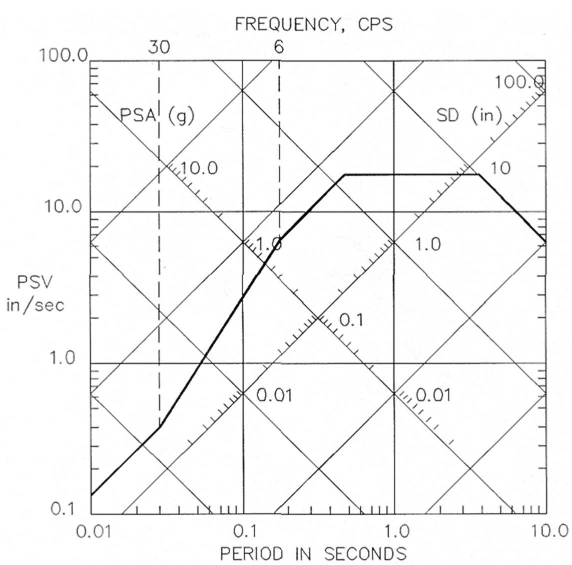

Figure 4-19. Response spectrum of Example 4-3.

4.4 GENERALIZED-COORDINATE APPROACH

Up to this point, the only structures which have been considered are single-story buildings which can be idealized as SDOF systems. The analysis of most structural systems requires a more complicated idealization even if the response can be represented in terms of a single degree of freedom. The generalized-coordinate approach provides a means of representing the response of more complex structural systems in terms of a single, time-dependent coordinate, known as the generalized coordinate.

Displacements in the structure are related to the generalized coordinate as

$$v(x,t) = \phi(x)Y(t) \tag{4-45}$$

Where $Y(t)$ is the time-dependent generalized coordinate and $\phi(x)$ is a spatial shape function which relates the structural degrees of freedom, $v(x, t)$, to the generalized coordinate. For a generalized SDOF system, it is necessary to represent the restoring forces in the damping elements and the stiffness elements in terms of the relative velocity and relative displacement between the ends of the element:

$$\Delta\dot{v}(x,t) = \Delta\phi(x)\dot{Y}(t) \tag{4-46}$$

$$\Delta v(x,t) = \Delta\phi(x)Y(t) \qquad \text{(4-47)}$$

Most structures can be idealized as a vertical cantilever, which limits the number of displacement functions that can be used to represent the horizontal displacement. Once the displacement function is selected, the structure is constrained to deform in that prescribed manner. This implies that the displacement functions must be selected carefully if a good approximation of the dynamic properties and response of the system are to be obtained. This section will develop the equations for determining the generalized response parameters in terms of the spatial displacement function and the physical response parameters. Methods for determining the shape function will be discussed, and techniques for determining the more correct displacement function for a particular structure will be presented.

4.4.1 Displacement Functions and Generalized Properties

Formulation of the equation of motion in terms of a generalized coordinate will be restricted to systems which consist of an assemblage of lumped masses and discrete elements. Lateral resistance is provided by discrete elements whose restoring force is proportional to the relative displacement between the ends of the element. Damping forces are proportional to the relative velocity between the ends of the discrete damping element. Formulation of the equation of motion for systems having distributed elasticity is described by Clough and Penzien. (4-4) The general equation of dynamic equilibrium is given in Equation 4-6, which represents a system of forces which are in equilibrium at any instant of time. The principle of virtual work in the form of virtual displacements states that

If a system of forces which are in equilibrium is given a virtual displacement which is consistent with the boundary conditions, the work done is zero.

Applying this principle to Equation 4-6 results in an equation of virtual work in the form

$$f_i \delta v + f_d \delta\Delta v + f_s \delta\Delta v - p(t)\,\delta v = 0 \qquad \text{(4-48)}$$

where it is understood that $v = v(x,t)$ and that the virtual displacements applied to the damping force and the elastic restoring force are virtual relative displacements. The virtual displacement can be expressed as

$$\delta v(x,t) = \phi(x)\delta Y(t) \qquad \text{(4-49)}$$

and the virtual relative displacement can be written as

$$\delta\Delta v(x,t) = \Delta\phi(x)\delta Y(t) \qquad \text{(4-50)}$$

where

$$\Delta v(x,t) = \phi(x_i)\,Y(t) - \phi(x_j)\,Y(t) = \Delta\phi(x)Y(t)$$

The inertia, damping and elastic restoring forces can be expressed as

$$\begin{aligned} f_i &= m\ddot{v} = m\phi\ddot{Y} \\ f_d &= c\Delta\dot{v} = c\Delta\phi\ddot{Y} \\ f_s &= k\Delta v = k\Delta\phi Y \end{aligned} \qquad \text{(4-51)}$$

Substituting Equations 4-49, 4-50, and 4-51 into Equation 4-48 results in the following equation of motion in terms of the generalized coordinate:

$$m^*\ddot{Y} + c^*\dot{Y} + k^*Y = p^*(t) \qquad \text{(4-52)}$$

where m^*, c^*, k^*, and p^* are referred to as the *generalized parameters* and are defined as

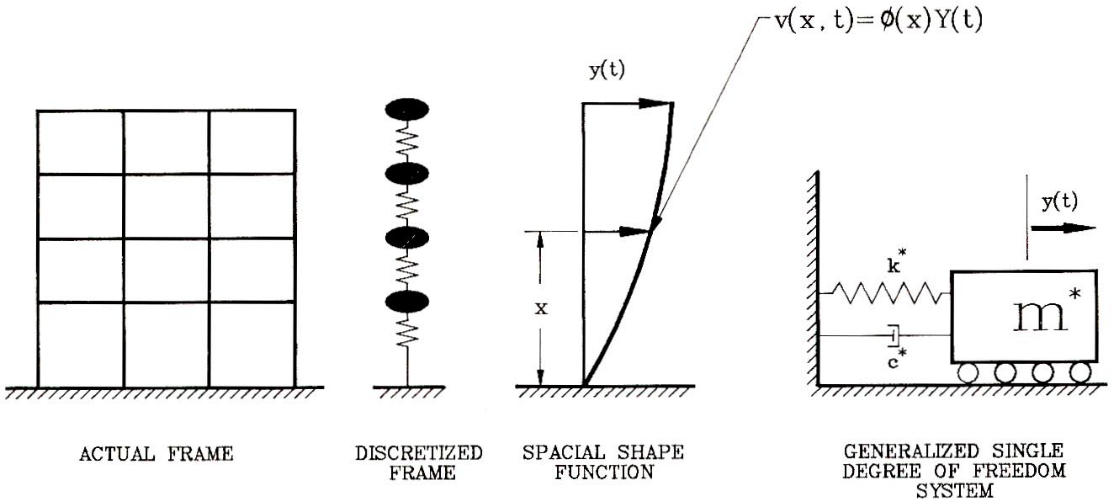

Figure 4-20. Generalized single-degree-of-freedom system.

$$
\begin{aligned}
m^* &= \sum_i m_i \phi_i^2 = \text{generalized mass} \\
c^* &= \sum_i c_i \Delta\phi_i^2 = \text{generalized damping} \\
k^* &= \sum_i k_i \Delta\phi_i^2 = \text{generalized stiffness} \\
p^* &= \sum_i p_i \phi_i = \text{generalized force}
\end{aligned}
\tag{4-53}
$$

For a time-dependent base acceleration the generalized force becomes

$$p^* = \ddot{g}\mathrm{L} \tag{4-54}$$

where

$$
\begin{aligned}
\mathrm{L} &= \sum_i m_i \phi_i \\
&= \text{earthquake participation factor}
\end{aligned}
\tag{4-55}
$$

It is also convenient to express the generalized damping in terms of the percent of critical damping in the following manner:

$$c^* = \sum_i c_i \Delta\phi(i)^2 = 2\lambda m^* \omega \tag{4-56}$$

Where ω represents the circular frequency of the generalized system and is given as

$$\omega = \sqrt{\frac{k^*}{m^*}} \tag{4-57}$$

The effect of the generalized-coordinate approach is to transform a multiple-degree-of-freedom dynamic system into an equivalent single-degree-of-freedom system in terms of the generalized coordinate. This transformation is shown schematically in Figure 4-20. The degree to which the response of the transformed system represents the actual system will depend upon how well the assumed displacement shape represents the dynamic displacement of the actual structure. The displacement shape depends on the aspect ratio of the structure, which is defined as the ratio of the height to the base dimension. Possible shape functions for high-rise, mid-rise, and low-rise structures are summarized in Figure 4-21. It should be noted that most building codes use the straight-line shape function which is shown for the mid-rise system. Once the dynamic response is obtained in terms of the generalized coordinate, Equation 4-45 must be used to determine the displacements in the structure, and these in turn

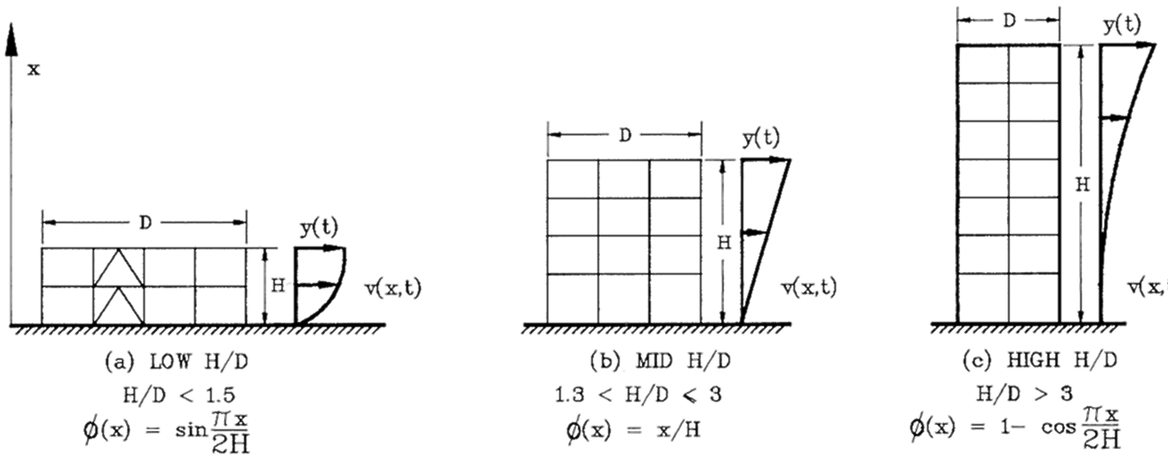

Figure 4-21. Possible shape functions based on aspect ratio.

can be used to determine the forces in the individual structural elements.

In principle, any function which represents the general deflection characteristics of the structure and satisfies the support conditions could be used. However, any shape other than the true vibration shape requires the addition of external constraints to maintain equilibrium. These extra constraints tend to stiffen the system and thereby increase the computed frequency. The true vibration shape will have no external constraints and therefore will have the lowest frequency of vibration. When choosing between several approximate deflected shapes, the one producing the lowest frequency is always the best approximation. A good approximation to the true vibration shape can be obtained by applying forces representing the inertia forces and letting the static deformation of the structure determine the spatial shape function.

Example 4-5 (Determination of generalized parameters)

Considering the four-story, reinforced-concrete moment frame building shown in Figure 4-22, determine the generalized mass, generalized stiffness, and fundamental period of vibration in the transverse direction using the following shape functions:

(a) $\phi(x) = \sin(\pi x / 2L)$ and (b) $\phi(x) = x / L$.All beams are 12in.$\times$20 in., and all columns are 14 in$\times$14 in. f_c'=4000 psi, and the modulus of elasticity of concrete is 3.6×10^6 psi. Reinforcing steel is made of grade-60 bars. Floor weights (total dead load) are assumed to be 390 kips at the roof, 445 kips at the fourth and third levels, and 448 kips at the first level. Live loads are 30 psf at the roof and 80 psf per typical floor level.

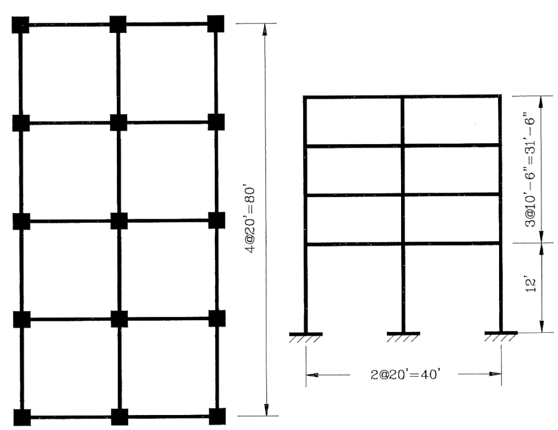

Figure 4-22. Building of Example 4-5.

Assuming beams are rigid relative to columns (Figure 4-23),

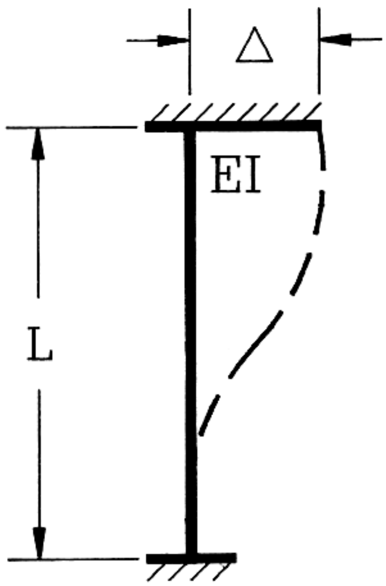

Figure 4-23. Assumed shape of column deformation.

$$V = \frac{12EI\Delta}{L^3}$$

$$K_i = \frac{V}{\Delta} = \frac{12EI}{L^3}$$

$$I_{col} = \frac{14(14)^3}{12} = 3201\,\text{in.}^4$$

$$I_{beam} = \frac{12(20)^3}{12} = 8000\,\text{in.}^4$$

$$K_{story} = \sum_{i=1}^{3} K_i = 3K_i \text{ (one frame)}$$

$$K_{4,3,2} = \frac{(3)(12)(3.6\times10^3)(3201)}{(126)^3} = 209\,\frac{\text{kips}}{\text{in.}}$$

$$K_1 = \frac{(3)(12)(3.6\times10^3)(3201)}{(144)^3} = 140\,\frac{\text{kips}}{\text{in.}}$$

Calculating generalized properties (see Figure 4-24):

(a) Assuming $\phi(x) = \sin(\pi x/2L)$:

Level	K	M	ϕ_i	$\Delta\phi_i$	$M\phi_i^2$	$K\Delta\phi_i^2$
4		0.252	1.000		0.252	
	209			0.071		1.054
3		0.288	0.929		0.249	
	209			0.203		8.613
2		0.288	0.726		0.152	
	209			0.306		19.570
1		0.290	0.420		0.051	
	140			0.420		24.696
					M* = 0.704	K* = 53.933

$$\omega = \sqrt{\frac{k^*}{m^*}} = \sqrt{\frac{53.93}{0.704}} = 8.75\,\text{rad/sec}$$
$$\text{and } T_a = 0.72\,\text{sec}$$

(b) Assuming $\phi(x) = x/L$

Level	K	M	ϕ_i	$\Delta\phi_i$	$M\phi_i^2$	$K\Delta\phi_i^2$
4		0.252	1.000		0.252	
	209			0.241		12.139
3		0.288	0.759		0.166	
	209			0.242		12.240
2		0.288	0.517		0.077	
	209			0.241		12.139
1		0.290	0.276		0.022	
	140			0.276		10.665
					M* = 0.517	K* = 47.183

$$\omega = \sqrt{\frac{k^*}{m^*}} = \sqrt{\frac{47.183}{0.517}}$$
$$= 9.55\,\text{rad/sec} \quad \text{and} \quad T_b = 0.66\,\text{sec}.$$

Since $T_a > T_b, \phi(x) = \sin(\pi x/2L)$ is a better approximation to the deflected shape than $\phi(x) = x/L$

4.4.2 Rayleigh's Method

Rayleigh's method is a procedure developed by Lord Rayleigh [(4-5)] for analyzing vibrating systems using the law of conservation of energy. Its principal use is for determining an accurate approximation of the natural frequency of a structure. The success of

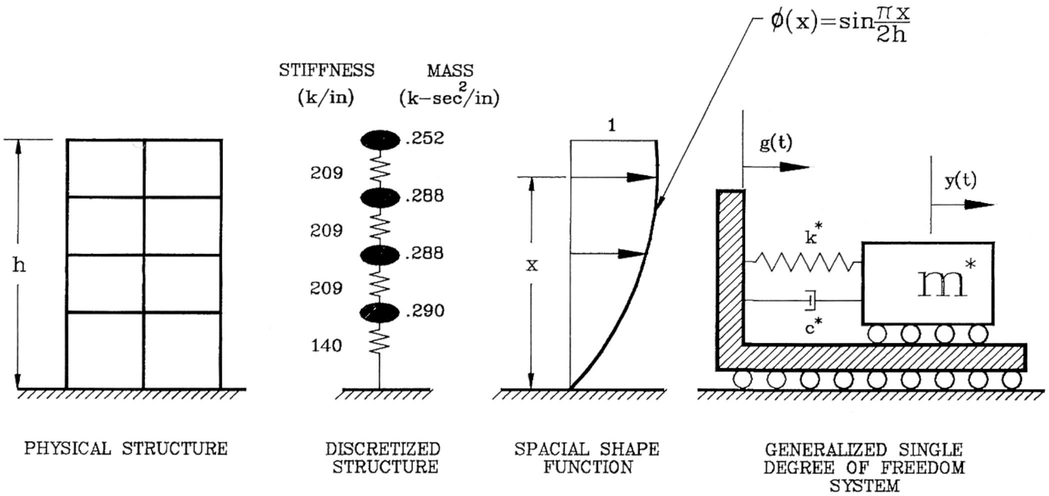

Figure 4-24. Development of a generalized SDOF model for building of Example *4*-4.

the technique in accomplishing this has been recognized by most building codes, which have adopted the procedure as an alternative for estimating the fundamental period of vibration. In addition to providing an estimate of the fundamental period, the procedure can also be used to estimate the shape function ϕ (x).

In an undamped elastic system, the maximum potential energy can be expressed in terms of the external work done by the applied forces. In terms of a generalized coordinate this expression can be written as

$$(PE)_{\max} = \frac{Y}{2}\sum p_i \phi_i = \frac{p^* Y}{2} \tag{4-58}$$

Similarly, the maximum kinetic energy can be expressed in terms of the generalized coordinate as

$$(KE)_{\max} = \frac{\omega^2 Y^2}{2}\sum_i m_i \phi_i^2 = \frac{\omega^2 Y^2 m^*}{2} \tag{4-59}$$

According to the principle of conservation of energy for an undamped elastic system, these two quantities must be equal to each other and to the total energy of the system. Equating Equation 4-58 to Equation 4-59 results in the following expression for the circular frequency:

$$\omega = \sqrt{\frac{p^*}{m^* Y}} \tag{4-60}$$

Substituting this result into Equation 4-20 for the period results in

$$T = 2\pi\sqrt{\frac{m^* Y}{p^*}} \tag{4-61}$$

Multiplying the numerator and denominator of the radical by Y and using Equation 4-45 results in the expression for the fundamental period:

$$T = 2\pi\sqrt{\frac{\sum_i w_i v_i^2}{g\sum p_i v_i}} \tag{4-62}$$

which is the expression found in most building codes.

The forces which must be applied laterally to obtain either the shape function ϕ (x) or the displacement *v(x)* represent the inertia forces, which are the product of the mass and the acceleration. If the acceleration is assumed to vary linearly over the height of a building with uniform weight distribution, a distribution of inertia force in the form of an inverted triangle will be obtained, being maximum at the top and zero at the bottom. This is similar to the distribution of base shear used in most building codes and can be a reasonable one to use when applying the Rayleigh method. The resulting deflections can be used directly in Equation 4-62 to estimate the period of vibration or they can be normalized in terms of the generalized coordinate (maximum displacement) to obtain the spatial shape function to be used in the generalized-coordinate method.

Example 4-6 (Application of Rayleigh's Method)

Use Rayleigh's method to determine the spatial shape function and estimate the fundamental period of vibration in the transverse direction for the reinforced-concrete building given in Example 4-4.

We want to apply static lateral loads that are representative of the inertial loads on the building. Since the story weights are approximately equal, it is assumed that the accelerations and hence the inertial loads vary linearly from the base to the roof (see Figure 4-25).

Note that the magnitude of loads is irrelevant and is chosen for ease of computation. The following computations (on the bottom of this page) are a tabular solution of Equation 4-61.

$$T = 2\pi\sqrt{\frac{m^*Y}{p^*}}, \quad \text{or}$$

$$T = 2\pi\sqrt{\frac{(0.666)(0.3343)}{16.912}} = 0.712 \text{ sec}$$

Note that since T = 0.721 is greater than either of the periods calculated in Example 4-5, the deflected shape given by applying the static loads is a better approximation than either of the two previous deflected shapes.

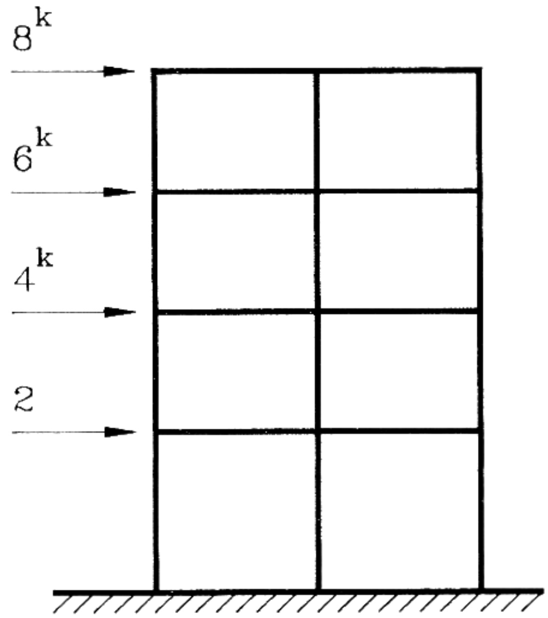

*Figure 4-25.*Frame of Example 4-5.

4.4.3 Earthquake Response of Elastic Structures

Time-History Analysis Substituting the generalized parameters of Equations 4-53 and 4-54 into the Duhamel-integral solution, Equation 4-33, results in the following solution for the displacement:

$$v(x,t) = \frac{\phi(x)\,\mathcal{L}\,V(t)}{m^*\omega} \tag{4-63}$$

Using Equation 4-37, the inertia force at any position x above the base can be obtained from

Level	K	m	P	V	Δ=V/k	v	ϕ	$m_i\phi_i^2$	$P_i\phi_i$
4		0.252	8.0			0.3343	1.000	0.252	8.000
	209			8	0.0383				
3		0.288	6.0			0.2960	0.886	0.226	5.316
	209			14	0.0670				
2		0.288	4.0			0.2290	0.685	0.135	2.740
	209			18	0.0861				
1		0.288	2.0			0.1429	0.428	0.053	0.856
	140			20	0.1429	0.000	0.000	0.666	16.912

$$q(x,t) = m(x)\ddot{v}(x,t) = m(x)\omega^2 v(x,t) \quad (4\text{-}64)$$

which, using Equation 4-63, becomes

$$q(x,t) = \frac{m(x)\phi(x)\mathcal{L}\ \omega V(t)}{m^*} \quad (4\text{-}65)$$

The base shear is obtained by summing the distributed inertia forces over the height H of the structure:

$$Q(t) = \int q(x,t)dx = \frac{\mathcal{L}^2}{m^*}\omega V(t) \quad (4\text{-}66)$$

The above relationships can be used to determine the displacements and forces in a generalized SDOF system at any time during the time history under consideration.

Response-Spectrum Analysis

The maximum value of the velocity given by Equation 4-35 is defined as the spectral pseudovelocity (S_{pv}), which is related to the spectral displacement (S_d) by Equation 4-43. Substituting this value into Equation 4-63 results in an expression for the maximum displacement in terms of the spectral displacement:

$$v(x)_{\max} = \frac{\phi(x)\ \mathcal{L}\ S_d}{m^*} \quad (4\text{-}67)$$

The forces in the system can readily be determined from the inertia forces, which can be expressed as

$$q(x)_{\max} = m(x)\ddot{v}(x)_{\max} = m(x)\omega^2 v(x)_{\max} \quad (4\text{-}68)$$

Rewriting this result in terms of the spectral pseudo-acceleration (S_{pa}) results in the following:

$$q(x)_{\max} = \frac{\phi(x)m(x)\mathcal{L}\ S_{pa}}{m^*} \quad (4\text{-}69)$$

Of considerable interest to structural engineers is the determination of the base shear. This is a key parameter in determining seismic design forces in most building codes. The base shear Q can be obtained from the above expression by simply summing the inertia forces and using Equation 4-55:

$$Q_{\max} = \frac{\mathcal{L}^2 S_{pa}}{m^*} \quad (4\text{-}70)$$

It is also of interest to express the base shear in terms of the effective weight, which is defined as

$$W^* = \frac{(\sum_i w_i \phi_i)^2}{\sum_i w_i \phi_i^2} \quad (4\text{-}71)$$

The expression for the maximum base shear becomes

$$Q_{\max} = W^* S_{pa} / g \quad (4\text{-}72)$$

This form is similar to the basic base-shear equation used in the building codes. In the code equation, the effective weight is taken to be equal to the total dead weight *W,* plus a percentage of the live load for special occupancies. The seismic coefficient *C* is determined by a formula but is equivalent to the spectral pseudoacceleration in terms of *g*. The basic code equation for base shear has the form

$$Q_{\max} = CW \quad (4\text{-}73)$$

The effective earthquake force can also be determined by distributing the base shear over the story height. This distribution depends upon the displacement shape function and has the form

$$q_i = Q_{\max}\frac{m_i\phi_i}{\mathcal{L}} \quad (4\text{-}74)$$

If the shape function is taken as a straight line, the code force distribution is obtained. The overturning moment at the base of the structure can be determined by multiplying the inertia force by the corresponding story height above the base and summing over all story levels:

$$M_O = \sum_i h_i q_i \tag{4-75}$$

Example 4-7 (Spectrum Analysis of Generalized SDOF System)

Using the design spectrum given in Figure 4-26, the shape function determined in Example 4-6, and the reinforced-concrete moment frame of Example 4-5, determine the base shear in the transverse direction, the corresponding distribution of inertia forces over the height of the structure, and the resulting overturning moment about the base of the structure.

$$T = 0.721 \text{ sec.}, \qquad f = 1/T = 1.39\,\text{Hz}, \qquad \omega = 8.715 \text{ rad/sec.}$$

From the design spectrum S_{pa} = 0.185g.

Level	m_i	ϕ_i	$m_i\phi_i^2$	$m_i\phi_i$	$m_i\phi_i/L$	q_{max}	V_{max}
4	0.252	1.000	0.252	0.252	0.305	27.10	
							27.10
3	0.288	0.866	0.226	0.255	0.308	27.36	
							54.46
2	0.288	0.685	0.135	0.197	0.238	21.14	
							75.60
1	0.288	0.428	0.053	0.123	0.149	13.24	
			0.666	0.827			88.84

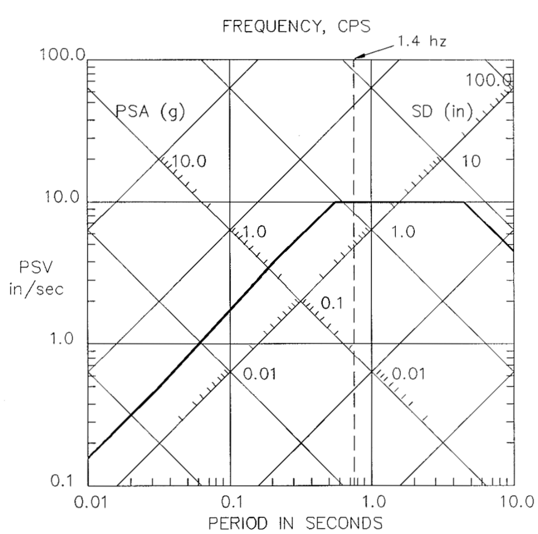

Figure 4-26. Design spectrum for Example 4-6.

From Equation 4-66,

$$Q_{\max} = \frac{(0.827)^2(0.185)(386.4)}{0.666} = 88.84 \text{ kips}$$

The overturning moment is: (see Fig, 4-27)

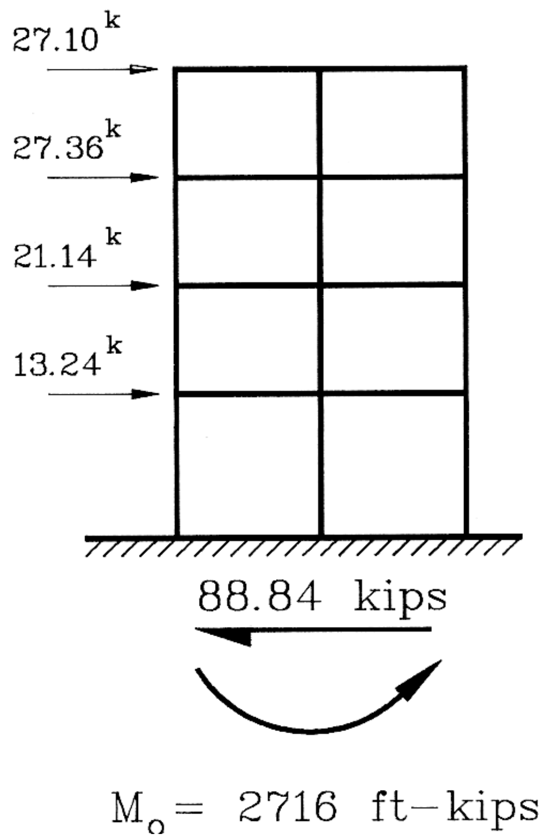

Figure 4-27. Story shears and overturning moment (Example *4*-6)

$$\begin{aligned} M_o &= 27.10(43.5) + 27.36(33) + \\ &\quad 21.14(22.5) + 13.24(12) \\ &= 2716 \text{ ft} - \text{kips} \end{aligned}$$

The displacement is

$$v_{\max} = \phi(\varphi / m^*)\, S_d = \phi\, \alpha\, S_d$$

where

$$S_d = S_{pa} / \omega^2 \qquad \text{and} \qquad \alpha = \varphi / m^*$$

$$S_d = \frac{(0.185)(386.4)}{(8.715)^2} = 0.941$$

$$\alpha = \frac{0.827}{0.666} = 1.242$$

$$v_i = (1.242)(0.941)\phi_i = 1.168\phi_i$$

$$v_4 = 1.168 \text{ in.} \qquad v_3 = 1.035 \text{ in.}$$

$$v_2 = 0.80 \text{ in.} \qquad v_1 = 0.50 \text{ in.}$$

4.5 RESPONSE OF NONLINEAR SDOF SYSTEMS

In an earlier section it was shown that the response of a linear structural system could be evaluated using the Duhamel integral. The approach was limited to linear systems because the Duhamel-integral approach makes use of the principle of superposition in developing the method. In addition, evaluation of the Duhamel integral for earthquake input motions will require the use of numerical methods in evaluating the integral. For these reasons it may be more expedient to use numerical integration procedures directly for evaluating the response of linear systems to general dynamic loading. These methods have the additional advantage that with only a slight modification they can be used to evaluate the dynamic response of nonlinear systems. Many structural systems will experience nonlinear response sometime during their life. Any moderate to strong earthquake will drive a structure designed by conventional methods into the inelastic range, particularly in certain critical regions. A very useful numerical integration technique for problems of structural dynamics is the so called step-by-step integration procedure. In this procedure the time history under consideration is divided into a number of small time increments Δt. During a small time step, the behavior of the structure is assumed to be linear. As nonlinear behavior occurs, the incremental stiffness is modified. In this manner, the response of the nonlinear system is approximated by a series of linear systems having a changing stiffness. The velocity and displacement computed at the end of one time interval become the initial conditions for the next time interval, and hence the process may be continued step by step.

4.5.1 Numerical Formulation of Equation of Motion

This section considers SDOF systems with properties *m, c, k(t)* and *p(t),* of which the applied force and the stiffness are functions of time. The stiffness is actually a function of the yield condition of the restoring force, and this in turn is a function of time. The damping coefficient may also be considered to be a function of time; however, general practice is to determine the damping characteristics for the elastic system and to keep these constant throughout the complete time history. In the inelastic range the principle mechanism for energy dissipation is through inelastic deformation, and this is taken into account through the hysteretic behavior of the restoring force.

The numerical equation required to evaluate the nonlinear response can be developed by first considering the equation of dynamic equilibrium given previously by Equation 4-6. It has been stated previously that this equation must be satisfied at every increment of time. Considering the time at the end of a short time step, Equation 4-6 can be written as

$$f_i(t+\Delta t) + f_d(t+\Delta t) + f_s(t+\Delta t) = p(t+\Delta t) \tag{4-76}$$

where the forces are defined as

$$f_i = m\ddot{v}(t+\Delta t)$$

$$f_d = c\dot{v}(t+\Delta t)$$

$$f_s = \sum_{i=1}^{n} k_i(t)\Delta v_i(t) = r_t + k(t)\Delta v(t) \tag{4-77}$$

$$\Delta v(t) = v(t+\Delta t) - v(t)$$

$$r_t = \sum_{i=1}^{n-1} k_i(t)\Delta v_i(t)$$

and in the case of ground accelerations

$$p(t+\Delta t) = p_e(t+\Delta t) = -m\ddot{g}(t+\Delta t) \quad (4\text{-}78)$$

Substituting Equations 4-77 and 4-78 into Equation 4-76 results in an equation of motion of the form

$$m\ddot{v}(t+\Delta t) + c\dot{v}(t+\Delta t) + \sum k_i \Delta v_i = -m\ddot{g}(t+\Delta t) \quad (4\text{-}79)$$

It should be noted that the incremental stiffness is generally defined by the tangent stiffness at the beginning of the time interval

$$k_i = \frac{df_s}{dv} \quad (4\text{-}80)$$

In addition, the dynamic properties given in Equations 4-77 and 4-78 can readily be exchanged for the generalized properties when considering a generalized SDOF system.

4.5.2 Numerical Integration

Many numerical integration schemes are available in the literature. The technique considered here is a step-by-step procedure in which the acceleration during a small time increment is assumed to be constant. A slight variation of this procedure, in which the acceleration is assumed to vary linearly during a small time increment, is described in detail by Clough and Penzien.[(4-4)]. Both procedures have been widely used and have been found to yield good results with minimal computational effort.

If the acceleration is assumed to be constant during the time interval, the equations for the constant variation of the acceleration, the linear variation of the velocity and the quadratic variation of the displacement are indicated in Figure 4-28. Evaluating the expression for velocity and displacement at the end of the time interval leads to the following two expressions for velocity and displacement:

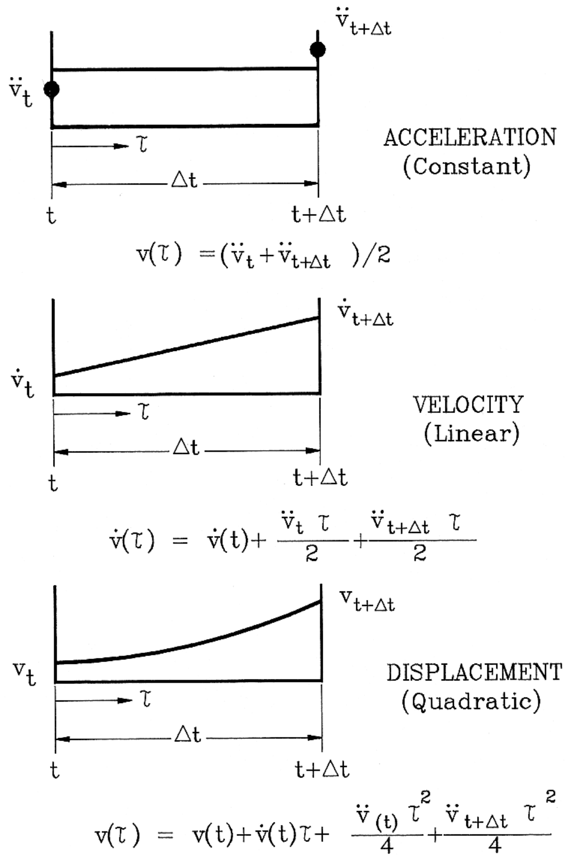

Figure 4-28. Increment motion (constant acceleration).

$$\dot{v}(t+\Delta t) = \dot{v}(t) + \ddot{v}(t+\Delta t)\frac{\Delta t}{2} + \ddot{v}(t)\frac{\Delta t}{2} \quad (4\text{-}81)$$

$$v(t+\Delta t) = v(t) + \dot{v}(t)\Delta t + \ddot{v}(t+\Delta t)\frac{\Delta t^2}{4} + \ddot{v}(t)\frac{\Delta t^2}{4} \quad (4\text{-}82)$$

Solving Equation 4-82 for the acceleration $\ddot{v}(t+\Delta t)$ gives

$$\ddot{v}(t+\Delta t) = \frac{4}{\Delta t^2}\Delta v - \frac{4}{\Delta t}\dot{v}(t) - \ddot{v}(t) \quad (4\text{-}83)$$

which can be written as

$$\ddot{v}(t+\Delta t) = \frac{4}{\Delta t^2}\Delta v + A(t) \quad (4\text{-}84)$$

where

$$\Delta v = v(t+\Delta t) - v(t)$$
$$A(t) = -\frac{4}{\Delta t}\dot{v}(t) - \ddot{v}(t)$$

Note that this equation expresses the acceleration at the end of the time interval as a function of the incremental displacement and the acceleration and velocity at the beginning of the time interval. Substituting Equation 4-83 into Equation 4-81 gives the following expression for the velocity at the end of the time increment:

$$\dot{v}(t+\Delta t) = \frac{2}{\Delta t}\Delta v - \dot{v}(t) \quad (4\text{-}85)$$

which can be written as

$$\dot{v}(t+\Delta t) = \frac{2}{\Delta t}\Delta v + B(t) \quad (4\text{-}86)$$

where

$$B(t) = -\dot{v}(t)$$

It is convenient to express the damping as a linear function of the mass:

$$c = \alpha m = \lambda C_{cr} = 2m\omega\lambda \quad (4\text{-}87)$$

Use of this equation allows the proportionality factor α to be expressed as

$$\alpha = 2\lambda\omega \quad (4\text{-}88)$$

Substituting Equations 4-85, 4-86, and 4-88 into Equation 4-79 results in the following form of the equation for dynamic equilibrium:

$$m\left[\frac{4}{\Delta t^2}\Delta v + A(t)\right] + \alpha m\left[\frac{2}{\Delta t}\Delta v + B(t)\right] + R(t) + k\Delta v$$
$$= m\ddot{g}(t+\Delta t) \quad (4\text{-}89)$$

Moving terms containing the response conditions at the beginning of the time interval to the right-hand side of the equation results in the following so-called pseudo-static form of the equation of motion:

$$\overline{k}_t(\Delta v) = \overline{p}(t+\Delta t) \quad (4\text{-}90)$$

where

$$\overline{k}_t = \frac{4m}{\Delta t^2} + \frac{2\alpha m}{\Delta t} + k_t$$
$$\overline{p}(t+\Delta t) = -m\ddot{g}(t+\Delta t) - R(t)$$
$$- m[A(t) - \alpha B(t)]$$

The solution procedure for a typical time step is as follows:

1. Given the initial conditions at the beginning of the time interval, calculate the coefficients A(t) and B(t).
2. Calculate the effective stiffness.
3. Determine the effective force.
4. Solve for the incremental displacement

$$v = \overline{p}/\overline{k}_t \quad (4\text{-}91)$$

5. Determine the displacement, velocity and acceleration at the end of the time interval:

$$v(t+\Delta t) = v(t) + \Delta v$$
$$\dot{v}(t+\Delta t) = \frac{2}{\Delta t} + B(t) \quad (4\text{-}92)$$
$$\ddot{v}(t+\Delta t) = \frac{4}{\Delta t^2} + A(t)$$

6. The values given in Equation 4-92 become the initial conditions for the next time increment, and the procedure is repeated.

The above algorithm can be easily programmed on any microcomputer. If it is combined with a data base of recorded earthquake data such as EQINFOS,[4-6] it can be used to gain considerable insight into the linear and nonlinear response of structures that can be modeled as either a **SDOF** system or as a generalized SDOF system. It also forms the background material for later developments for multiple-degree-of-freedom systems.

An important response parameter that is unique to nonlinear systems is the ductility ratio. For a SDOF system, this parameter can be defined in terms of the displacement as

$$\mu = \frac{v(\max)}{v(\text{yield})} = 1.0 + \frac{v(\text{plastic})}{v(\text{yield})} \qquad (4\text{-}93)$$

As can be seen from the above equation, the ductility ratio is an indication of the amount of inelastic deformation that has occurred in the system. In the case of a SDOF system or generalized SDOF system the ductility obtained from Equation 4-93 usually represents the average ductility in the system. The ductility demand at certain critical regions, such as plastic hinges in critical members, may be considerably higher.

4.6 MULTIPLE-DEGREE-OF-FREEDOM SYSTEMS

In many structural systems it is impossible to model the dynamic response accurately in terms of a single displacement coordinate. These systems require a number of independent displacement coordinates to describe the displacement of the mass of the structure at any instant of time.

4.6.1 Mass and Stiffness Properties

In order to simplify the solution it is usually assumed for building structures that the mass of the structure is lumped at the center of mass of the individual story levels. This results in a diagonal matrix of mass properties in which either the translational mass or the mass moment of inertia is located on the main diagonal.

$$\{f_i\} = \begin{bmatrix} m_1 & & & & & & \\ & m_2 & & & & & \\ & & m_3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & m_n \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{Bmatrix} \qquad (4\text{-}94)$$

It is also convenient for building structures to develop the structural stiffness matrix in terms of the stiffness matrices of the individual story levels. The simplest idealization for a multistory building is based on the following three assumptions: (i) the floor diaphragm is rigid in its own plane; (ii) the girders are rigid relative to the columns and (iii) the columns are flexible in the horizontal directions but rigid in the vertical. If these assumptions are used, the building structure is idealized as having three dynamic degrees of freedom at each story level: a translational degree of freedom in each of two orthogonal directions, and a rotation about a vertical axis through the center of mass. If the above system is reduced to a plane frame, it will have one horizontal translational degree of freedom at each story level. The stiffness matrix for this type of structure has the tridiagonal form shown below:

For the simplest idealization, in which each story level has one translational degree of freedom, the stiffness terms k_i in the above equations represent the translational story stiffness of the ith story level. As the assumptions given above are relaxed to include axial deformations in the columns and flexural deformations in the girders, the stiffness term k_i in Equation 4-95 becomes a submatrix of stiffness terms, and the story displacement v_i

$$\{f_s\} = \begin{bmatrix} k_1 & -k_2 & & & & & & \\ k_2 & k_1+k_2 & -k_3 & & & & & \\ & -k_3 & k_2+k_3 & -k_4 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & -k_n \\ & & & & & & -k_n & k_{n-1}+k_n \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \\ v_n \end{Bmatrix} \qquad (3-95)$$

becomes a subvector containing the various displacement components in the particular story level. The calculation of the stiffness coefficients for more complex structures is a standard problem of static structural analysis. For the purposes of this chapter it will be assumed that the structural stiffness matrix is known.

4.6.2 Mode Shapes and Frequencies

The equations of motion for undamped free vibration of a multiple-degree-of-freedom (MDOF) system can be written in matrix form as

$$[M]\{\ddot{v}\} + [K]\{v\} = \{0\} \qquad \text{(4-96)}$$

Since the motions of a system in free vibration are simple harmonic, the displacement vector can be represented as

$$\{v\} = \{\bar{v}\}\sin \omega t \qquad \text{(4-97)}$$

Differentiating twice with respect to time results in

$$\{\ddot{v}\} = -\omega^2\{v\} \qquad \text{(4-98)}$$

Substituting Equation 4-98 into Equation 4-96 results in a form of the eigenvalue equation,

$$\left([K] - \omega^2[M]\right)\{v\} = \{0\} \qquad \text{(4-99)}$$

The classical solution to the above equation derives from the fact that in order for a set of homogeneous equilibrium equations to have a nontrivial solution, the determinant of the coefficient matrix must be zero:

$$\det([K] - \omega^2[M]) = \{0\} \qquad \text{(4-100)}$$

Expanding the determinant by minors results in a polynomial of degree *N*, which is called the frequency equation. The *N* roots of the polynomial represent the frequencies of the *N* modes of vibration. The mode having the lowest frequency (longest period) is called the first or fundamental mode. Once the frequencies are known, they can be substituted one at a time into the equilibrium Equation 4-99, which can then be solved for the relative amplitudes of motion for each of the displacement components in the particular mode of vibration. It should be noted that since the absolute amplitude of motion is indeterminate, N-1 of the displacement components are determined in terms of one arbitrary component.

This method can be used satisfactorily for systems having a limited number of degrees of freedom. Programmable calculators have programs for solving the polynomial equation and for doing the matrix operations required to determine the mode shapes. However, for problems of any size, digital computer programs which use numerical techniques to solve large eigenvalue systems(4-7) must be used.

Example 4-8 (Mode Shapes and Frequencies)

It is assumed that the response in the transverse direction for the reinforced-concrete moment frame of Example 4-4 can be represented in terms of four displacement degrees of freedom which represent the horizontal displacements of the four story levels. Determine the stiffness matrix and the mass matrix, assuming that the mass is lumped at the story levels. Use these properties to calculate the frequencies and mode shapes of the four-degree-of-freedom system.

•*Stiffness and mass matrices:* The stiffness coefficient k_{ij} is defined as the force at coordinate i due to a unit displacement at coordinate j, all other displacements being zero (see Figure 4-29):

$$[K] = \begin{bmatrix} 209 & -209 & 0 & 0 \\ -209 & 418 & -209 & 0 \\ 0 & -209 & 418 & -209 \\ 0 & 0 & -209 & 349 \end{bmatrix}$$

$$[M] = \frac{1}{4}\begin{bmatrix} 1.01 & 0 & 0 & 0 \\ 0 & 1.15 & 0 & 0 \\ 0 & 0 & 1.15 & 0 \\ 0 & 0 & 0 & 1.16 \end{bmatrix}$$

$$[K]-\omega^2[M] = 200\begin{bmatrix} 1.05-1.01B & -1.05 & 0 & 0 \\ -1.05 & 2.09-1.15B & -1.05 & 0 \\ 0 & -1.05 & 2.09-1.15B & -1.05 \\ 0 & 0 & -1.05 & 1.74-1.16B \end{bmatrix}$$

where $B = \omega^2/800$

•*Characteristic equation:*

$$\left|[K]-\omega^2[M]\right| = 0$$

$$B^4 - 6.183B^3 + 11.476B^2 - 6.430B + 0.486 = 0$$

Solution:

$$B_1 = 0.089 = \frac{\omega_1^2}{800},\ \omega_1 = 8.438,\quad T_1 = 0.744\,\text{sec}$$

$$B_2 = 0.830 = \frac{\omega_2^2}{800},\ \omega_2 = 25.768,\ T_1 = 0.244\,\text{sec}$$

$$B_3 = 2.039 = \frac{\omega_3^2}{800},\ \omega_3 = 40.388,\ T_3 = 0.155\,\text{sec}$$

$$B_4 = 3.225 = \frac{\omega_4^2}{800},\ \omega_4 = 50.800,\ T_4 = 0.124\,\text{sec}$$

•*Mode shapes* (see Figure 4-29) are obtained by substituting the values of B_i, one at a time, into the equations

$$([K]-\omega^2[M])\{v\} = \{0\}$$

and determining N-1 components of the displacement vector in terms of the first component, which is set equal to unity. This results in the modal matrix

$$[\Phi] = \begin{bmatrix} 1.00 & 1.00 & 1.00 & 1.00 \\ 0.91 & 0.20 & -1.07 & -1.78 \\ 0.74 & -0.78 & -0.75 & 1.75 \\ 0.47 & -1.05 & 1.24 & -0.92 \end{bmatrix}$$

Solution of the above problem using the computer program ETABS (4-12) gives the following results:

$$\{T\} = \begin{Bmatrix} 0.838 \\ 0.268 \\ 0.152 \\ 0.107 \end{Bmatrix}$$

$$[\Phi] = \begin{bmatrix} 1.00 & 1.00 & 1.00 & 1.00 \\ 0.91 & 0.20 & -1.07 & -1.78 \\ 0.74 & -0.78 & 0.75 & 1.75 \\ 0.47 & -1.05 & 1.24 & -0.92 \end{bmatrix}$$

This program assumes the floor diaphragm is rigid in its own plane but allows axial deformation in the columns and flexural deformations in the beams. Hence, with these added degrees of freedom (fewer constraints) the fundamental period increases. However, comparing the results of this example with those of Example 4-5, it can be seen that for this structure a good approximation for the first-mode response was obtained using the generalized SDOF model and the static deflected shape.

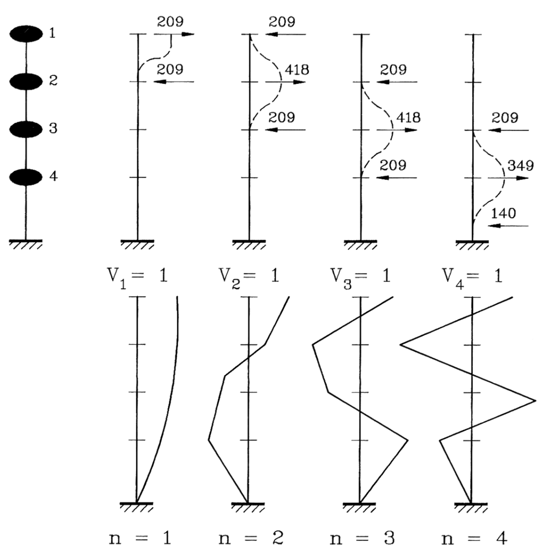

Figure 4-29. Stiffness determination and mode shape(Example 4-8).

4.6.3 Equations of Motion in Normal Coordinates

Betti's reciprocal work theorem can be used to develop two orthogonality properties of vibration mode shapes which make it possible to greatly simplify the equations of motion. The first of these states that the mode shapes are orthogonal to the mass matrix and is expressed in matrix form as

$$\{\phi_n\}^T [M] \{\phi_m\} = \{0\} \qquad (m \neq n) \quad (4\text{-}101)$$

Using Equations 4-99 and 4-101, the second property can be expressed in terms of the stiffness matrix as

$$\{\phi_n\}^T [K] \{\phi_m\} = \{0\} \qquad (m \neq n) \quad (4\text{-}102)$$

which states that the mode shapes are orthogonal to the stiffness matrix. It is further assumed that the mode shapes are also orthogonal to the damping matrix:

$$\{\phi_n\}^T [C] \{\phi_m\} = \{0\} \qquad (m \neq n) \qquad (4\text{-}103)$$

Sufficient conditions for this assumption have been discussed elsewhere.(4-8) Since any MDOF system having N degrees of freedom also has N independent vibration mode shapes, it is possible to express the displaced shape of the structure in terms of the amplitudes of these shapes by treating them as generalized coordinates (sometimes called normal coordinates). Hence the displacement at a particular location, v_i, can be obtained by summing the contributions from each mode as

$$v_i = \sum_{n=1}^{N} \phi_{in} Y_n \qquad (4\text{-}104)$$

In a similar manner, the complete displacement vector can be expressed as

$$\{v\} = \sum_{n=1}^{N} \{\phi_n\} Y_n = [\Phi]\{Y\} \tag{4-105}$$

It is convenient to write the equations of motion for a MDOF system in matrix form as

$$[M]\{\ddot{v}\} + [C]\{\dot{v}\} + [K]\{v\} = \{P(t)\} \tag{4-106}$$

which is similar to the equation for a SDOF system, Equation 4-9. The differences arise because the mass, damping, and stiffness are now represented by matrices of coefficients representing the added degrees of freedom, and the acceleration, velocity, displacement, and applied load are represented by vectors containing the additional degrees of freedom. The equations of motion can be expressed in terms of the normal coordinates by substituting Equation 4-105 and its appropriate derivatives into Equation 4-106 to give

$$[M][\Phi]\{\ddot{Y}\} + [C][\Phi]\{\dot{Y}\} + [K][\Phi]\{Y\} = \{P(t)\} \tag{4-107}$$

Multiplying the above equation by the transpose of any modal vector $\{\phi_n\}$ results in the following:

$$\begin{aligned}&\{\phi_n\}^T [M][\Phi]\{\ddot{Y}\} + \{\phi_n\}^T [C][\Phi]\{\dot{Y}\} \\ &+ \{\phi_n\}^T [K][\Phi]\{Y\} = \{\phi_n\}^T \{P(t)\}\end{aligned} \tag{4-108}$$

Using the orthogonality conditions of Equations 4-101, 4-102, and 4-103 reduces this set of equations to the equation of motion for a generalized SDOF system in terms of the generalized properties for the n th mode shape and the normal coordinate Y_n:

$$M_n^* \ddot{Y}_n + C_n^* \dot{Y}_n + K_n^* Y = P_n^*(t) \tag{4-109}$$

where the generalized properties for the nth mode are given as

$$\begin{aligned}
M_n^* &= \text{generalized mass} = \{\phi_n\}^T [M]\{\phi_n\} \\
C_n^* &= \text{generalized damping} \\
&= \{\phi_n\}^T [C]\{\phi_n\} = 2\lambda_n \omega_n M_n^* \\
K_n^* &= \text{generalized stiffness} \\
&= \{\phi_n\}^T [K]\{\phi_n\} = \omega_n^2 M_n^* \\
P_n^*(t) &= \text{generalized loading} = \{\phi_n\}^T \{P(t)\}
\end{aligned} \tag{4-110}$$

The above relations can be used to further simplify the equation of motion for the nth mode to the form

$$\ddot{Y}_n + 2\lambda_n \omega_n Y_n + \omega_n^2 Y_n = \frac{P_n^*(t)}{M_n^*} \tag{4-111}$$

The importance of the above transformations to normal coordinates has been summarized by Clough and Penzien,[(4-4)] who state that

The use of normal coordinates serves to transform the equations of motion from a set of N simultaneous differential equations which are coupled by off diagonal terms in the mass and stiffness matrices to a set of N independent normal coordinate equations.

It should further be noted that the expressions for the generalized properties of any mode are equivalent to those defined previously for a generalized SDOF system. Hence the use of the normal modes transforms the MDOF system having N degrees of freedom into a system of N independent generalized SDOF systems. The complete solution for the system is then obtained by superimposing the independent modal solutions. For this reason this method is often referred to as the modal-superposition method. Use of this method also leads to a significant saving in computational effort, since in most cases it will not be necessary to use all N modal responses to accurately represent the response of the structure. For most structural systems the lower modes make the primary contribution to the total response. Therefore, the response can usually be represented to sufficient accuracy in terms of a limited number of modal responses in the lower modes.

4.6.4 Earthquake-Response Analysis

Time-History Analysis As in the case of SDOF systems, for earthquake analysis the time-dependent force must be replaced with the effective loads, which are given by the product of the mass at any level, *M,* and the ground acceleration *g(t).* The vector of effective loads is obtained as the product of the mass matrix and the ground acceleration:

$$P_e(t) = [M]\{\Gamma\}\ddot{g}(t) \tag{4-112}$$

where $\{\Gamma\}$ is a vector of influence coefficients of which component i represents the acceleration at displacement coordinate i due to a unit ground acceleration at the base. For the simple structural model in which the degrees of freedom are represented by the horizontal displacements of the story levels, the vector $\{\Gamma\}$ becomes a unity vector, $\{1\}$, since for a unit ground acceleration in the horizontal direction all degrees of freedom have a unit horizontal acceleration. Using Equation 4-108, the generalized effective load for the *n*th mode is given as

$$P^*_{en}(t) = \mathrm{L}_n g(t) \tag{4-113}$$

Where $\mathrm{L}_n = \{\phi_n\}^T[M]\{\Gamma\}$

Substituting Equation 4-113 into Equation 4-111 results in the following expression for the earthquake response of the nth mode of a MDOF system:

$$\ddot{Y}_n + 2\lambda_n\omega_n\dot{Y}_n + \omega_n^2 Y_n = \varphi_n\ddot{g}(t)/M_n^* \tag{4-114}$$

In a manner similar to that used for the SDOF system, the response of this mode at any time t can be obtained by the Duhamel integral expression

$$Y_n(t) = \frac{\varphi_n V_n(t)}{M_n^*\omega_n} \tag{4-115}$$

where $V_n(t)$ represents the integral

$$V_n(t) = \int_0^t \ddot{g}(\tau)e^{-\lambda_n\omega_n(t-\tau)}\sin\omega_n(t-\tau)d\tau \tag{4-116}$$

The complete displacement of the structure at any time is then obtained by superimposing the contributions of the individual modes using Equation 4-105

$$\{v(t)\} = \sum_{n=1}^{N}\{\phi_n\}Y_n(t) = [\Phi]\{Y(t)\} \tag{4-117}$$

The resulting earthquake forces can be determined in terms of the effective accelerations, which for each mode are given by the product of the circular frequency and the displacement amplitude of the generalized coordinate:

$$\ddot{Y}_{ne}(t) = \omega_n^2 Y_n(t) = \frac{\varphi_n\omega_n V_n(t)}{M_n^*} \tag{4-118}$$

The corresponding acceleration in the structure due to the n th mode is given as

$$\{\ddot{v}_{ne}(t)\} = \{\phi_n\}\ddot{Y}_{ne}(t) \tag{4-119}$$

and the corresponding effective earthquake force is given as

$$\begin{aligned}\{q_n(t)\} &= [M]\{\ddot{v}_n(t)\} \\ &= [M]\{\phi_n\}\omega_n\varphi_n V_n(t)/M_n^*\end{aligned} \tag{4-120}$$

The total earthquake force is obtained by superimposing the individual modal forces to obtain

$$q(t) = \sum_{n=1}^{N} q_n(t) = [M][\Phi]\omega^2 Y(t) \quad (4\text{-}121)$$

The base shear can be obtained by summing the effective earthquake forces over the height of the structure:

$$\begin{aligned} Q_n(t) &= \sum_{i=1}^{H} q_{in}(t) = \{1\}^T \{q_n(t)\} \\ &= M_{en}\omega_n V_n(t) \end{aligned} \quad (4\text{-}122)$$

where $M_{en} = \text{L}_n^2 / M_n^*$ is the effective mass for the *n*th mode.

The sum of the effective masses for all of the modes is equal to the total mass of the structure. This results in a means of determining the number of modal responses necessary to accurately represent the overall structural response. If the total response is to be represented in terms of a finite number of modes and if the sum of the corresponding modal masses is greater than a predefined percentage of the total mass, the number of modes considered in the analysis is adequate. If this is not the case, additional modes need to be considered. The base shear for the nth mode, Equation 4-122, can also be expressed in terms of the effective weight, W_{en}, as

$$Q_n(t) = \frac{W_{en}}{g}\omega_n V_n(t) \quad (4\text{-}123)$$

where

$$W_{en} = \frac{\left(\sum_{i=1}^{H} W_i \phi_{in}\right)^2}{\sum_{i=1}^{H} W_i \phi_{in}^2} \quad (4\text{-}124)$$

The base shear can be distributed over the height of the building in a manner similar to Equation 4-74, with the modal earthquake forces expressed as

$$\{q_n(t)\} = \frac{[M]\{\phi_n\}Q_n(t)}{\text{L}_n} \quad (4\text{-}125)$$

4.6.5 Response-Spectrum Analysis

The above equations for the response of any mode of vibration are exactly equivalent to the expressions developed for the generalized SDOF system. Therefore, the maximum response of any mode can be obtained in a manner similar to that used for the generalized SDOF system. By analogy to Equations 4-34 and 4-43 the maximum modal displacement can be written as

$$Y_n(t)_{\max} = \frac{V_n(t)_{\max}}{\omega_n} = S_{dn} \quad (4\text{-}126)$$

Making this substitution in Equation 4-115 results in

$$Y_{n\max} = \varphi_n S_{dn} / M_n^* \quad (4\text{-}127)$$

The distribution of the modal displacements in the structure can be obtained by multiplying this expression by the modal vector

$$\{v_n\}_{\max} = \{\phi_n\} Y_{n\max} = \frac{\{\phi_n\}\text{L}_n S_{dn}}{M_n^*} \quad (4\text{-}128)$$

The maximum effective earthquake forces can be obtained from the modal accelerations as given by Equation 4-120:

$$\{q_n\}_{\max} = \frac{[M]\{\phi_n\}\varphi_n S_{pan}}{M_n^*} \quad (4\text{-}129)$$

Summing these forces over the height of the structure gives the following expression for the maximum base shear due to the *n*th mode:

$$Q_{n\max} = \varphi_n^2 S_{pan} / M_n^* \quad (4\text{-}130)$$

which can also be expressed in terms of the effective weight as

$$Q_{n\,\max} = W_{en} S_{pan} / g \tag{4-131}$$

where W_{en} is defined by Equation 4-124.

Finally, the overturning moment at the base of the building for the nth mode can be determined as

$$M_o = \langle h \rangle [M] \{\phi_n\} \mathrm{L}_n S_{pan} / M_n^* \tag{4-132}$$

where $\langle h \rangle$ is a row vector of the story heights above the base.

4.6.6 Modal Combinations

Using the response-spectrum method for MDOF systems, the maximum modal response is obtained for each mode of a set of modes, which are used to represent the response. The question then arises as to how these modal maxima should be combined in order to get the best estimate of the maximum total response. The modal-response equations such as Equations 4-117 and 4-121 provide accurate results only as long as they are evaluated concurrently in time. In going to the response-spectrum approach, time is taken out of these equations and replaced with the modal maxima. These maximum response values for the individual modes cannot possibly occur at the same time; therefore, a means must be found to combine the modal maxima in such a way as to approximate the maximum total response. One such combination that has been used is to take the sum of the absolute values (SAV) of the modal responses. This combination can be expressed as

$$r \le \sum_{n=1}^{N} |r_n| \tag{4-133}$$

Since this combination assumes that the maxima occur at the same time and that they also have the same sign, it produces an upper-bound estimate for the response, which is too conservative for design application. A more reasonable estimate, which is based on probability theory, can be obtained by using the square-root-of-the-sum-of-the-squares (SRSS) method, which is expressed as

$$r \approx \sqrt{\sum_{n=1}^{N} r_n^2} \tag{4-134}$$

This method of combination has been shown to give a good approximation of the response for two-dimensional structural systems. For three-dimensional systems, **it** has been shown that the complete-quadratic-combination (CQC) method [(4-9)] may offer a significant improvement in estimating the response of certain structural systems. The complete quadratic combination is expressed as

$$r \approx \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} r_i p_{ij} r_j} \tag{4-135}$$

where for constant modal damping

$$p_{ij} = \frac{8\lambda^2 (1+\zeta)\zeta^{3/2}}{(1-\zeta^2)^2 + 4\lambda^2 \zeta (1+\zeta)^2} \tag{4-136}$$

and

$$\zeta = \omega_j / \omega_i$$
$$\lambda = c / c_{cr}$$

Using the SRSS method for two-dimensional systems and the CQC method for either two- or three-dimensional systems will give a good approximation to the maximum earthquake response of an elastic system without requiring a complete time-history analysis. This is particularly important for purposes of design.

Table 3-3. Computation of response for model of Example *4*-8

Response Quantity	Modal Parameter	Modal Response n = 1	2	3	4	Combined Response SAV	SRSS	CQC
	$\omega =$	8.44	25.77	40.39	50.80			
	$\alpha_n =$	1.212	-0.289	0.075	0.010			
	$S_d =$	1.190	0.155	0.062	0.039			
	$\phi =$	*1.00*	*1.00*	*1.00*	*1.00*			
		0.91	*0.20*	*-1.07*	*-1.78*			
		0.74	*-0.78*	*-0.75*	*1.75*			
		0.47	*-1.05*	*1.24*	*-0.92*			
Displacement	n = 4	1.44	-0.045	0.019	-0.002	1.506	1.441	1.441
$v_n=\phi_n\alpha_n S_{dn}$	3	1.31	-0.009	-0.020	0.003	1.342	1.310	1.310
(Eq.3.128)	2	1.07	0.035	-0.014	-0.003	1.122	1.071	1.071
	1	0.68	0.047	0.023	0.001	0.751	0.682	0.682
Acceleration	n= 4	102.6	-29.9	31.0	-5.1	168.6	111.4	110.7
$\ddot{v}_n = \omega_n^2 v_n$	3	93.3	-6.0	-32.6	7.7	139.6	99.3	98.9
	2	76.2	23.2	-22.8	-7.7	129.9	83.2	83.3
	1	48.4	31.2	37.5	2.6	119.7	68.8	70.0
Inertia force	n = 4	25.91	-7.54	7.83	-1.30	42.6	28.1	27.9
$q_n=M\ddot{v}_n$	3	26.82	-1.72	-9.38	2.23	40.2	28.6	28.4
	2	21.91	6.68	-6.56	-2.23	37.4	23.9	23.9
	1	14.03	9.05	11.35	0.75	35.2	20.2	20.6
Shear	n= 4	25.91	-7.54	7.83	-1.30	42.6	28.1	28.0
$Q_n=\Sigma q_n$	3	52.73	-9.26	-1.55	0.93	64.5	53.6	53.5
	2	74.64	-2.58	-8.11	-1.30	86.6	75.1	75.1
	1	88.67	6.47	3.24	-0.55	98.9	89.0	89.0
Overturning	n= 4	272.1	-79.2	82.2	-13.7	447.2	295.4	293.6
Moment	3	825.7	-176.4	65.9	-3.9	1071.9	846.9	845.3
(ft-kips)	2	1609.4	-203.5	-19.2	-17.5	1849.6	1622.4	1621.3
	1	2673.4	-125.9	19.7	-24.1	2843.1	2676.5	2675.7

Example 4-9 (Response Spectrum Analysis)

Use the design response spectrum given in Example 4-7 and the results of Example 4-8 to perform a response-spectrum analysis of the reinforced concrete frame. Determine the modal responses of the four modes of vibration, and estimate the total response using the SAV, SRSS, and CQC methods of modal combination. Present the data in a tabular form suitable for hand calculation. Finally, compare the results with those obtained in Example 4-6 for a generalized SDOF model.

From Example 4-7,

$$[M] = \frac{1}{4}\begin{bmatrix} 1.01 & 0 & 0 & 0 \\ 0 & 1.15 & 0 & 0 \\ 0 & 0 & 1.15 & 0 \\ 0 & 0 & 0 & 1.16 \end{bmatrix}$$

$$\{\omega\} = \begin{Bmatrix} 8.44 \\ 25.77 \\ 40.39 \\ 50.80 \end{Bmatrix} \frac{r}{\text{sec}}$$

$$[\Phi] = \begin{bmatrix} 1.00 & 1.00 & 1.00 & 1.00 \\ 0.91 & 0.20 & -1.07 & -1.78 \\ 0.74 & -0.78 & -0.75 & 1.75 \\ 0.47 & -1.05 & 1.24 & -0.92 \end{bmatrix}$$

$$\{f\} = \frac{\omega}{2\pi} = \begin{Bmatrix} 1.34 \\ 4.10 \\ 6.43 \\ 8.09 \end{Bmatrix} \text{Hz}$$

$$S_v = \begin{Bmatrix} 10.0 \\ 4.0 \\ 2.5 \\ 2.0 \end{Bmatrix} \text{in./sec}$$

$$S_{dn} = S_{vn} / \omega_n$$

From Equation 4-128,

$$\{v_n\}_{\max} = \{\phi_n\}(\varphi_n / M_n^*) S_{dn} = \{\phi\}\alpha S_{dn}$$

$$\{q_n\} = [M]\{\ddot{v}_n\} = [M]\omega^2\{v_n\}$$

$$Q_n = \sum_{i=1}^{N} q_{ni}$$

For CQC combination,
$\lambda = 0.05$ = constant for all modes

$$p_{ij} = \begin{bmatrix} 1.0000 & 0.0062 & .0025 & .0017 \\ 0.0062 & 1.0000 & 0.0452 & 0.0193 \\ 0.0025 & 0.0452 & 1.0000 & 0.1582 \\ 0.0017 & 0.0193 & 0.1582 & 1.0000 \end{bmatrix}$$

The computation of the modal and the combined response is tabulated in Table 4-3. The results are compared with those obtained for the SDOF model in Table 4-4.

Table 4-4. Comparison of results obtained from MDOF and SDOF models.

Response parameter	MDOF (Example 3-9)	SDOF (Example 3-7)
Period (sec)	0.744	0.721
Displacements(in)		
Roof	1.44	1.17
3rd	1.31	1.04
2nd	1.07	0.80
1st	0.68	0.50
Inertia force (kips)		
Roof	28.1	27.1
3rd	28.6	27.4
2nd	23.9	21.1
1st	20.2	13.2
Base shear (kips)	89.0	88.8
Overturning moment (ft-kips)	2678	2716

4.7 NONLINEAR RESPONSE OF MDOF SYSTEMS

The nonlinear analysis of buildings modeled as multiple degree of freedom systems (MDOF) closely parallels the development for single degree of freedom systems presented earlier. However, the nonlinear dynamic time history analysis of MDOF systems is currently considered to be too complex for general use. Therefore, recent developments in the seismic evaluation of buildings have suggested a performance-based procedure which requires the determination of the demand and capacity. Demand is represented by the earthquake ground motion and its effect on a particular structural system. Capacity is the structure's ability to resist the seismic demand. In order to estimate the structure's capacity beyond the elastic limit, a static nonlinear (pushover) analysis is recommended [4-17]. For more demanding investigations of building response, nonlinear dynamic analyses can be conducted.

For dynamic analysis the loading time history is divided into a number of small time increments, whereas, in the static analysis, the lateral force is divided into a number of small force increments. During a small time or force increment, the behavior of the structure is assumed to be linear elastic. As nonlinear behavior occurs, the incremental stiffness is modified for the next time (load) increment. Hence, the response of the nonlinear system is approximated by the response of a sequential series of linear systems having varying stiffnesses.

Static Nonlinear Analysis

Nonlinear static analyses are a subset of nonlinear dynamic analyses and can use the same solution procedure without the time related inertia forces and damping forces. The equations of equilibrium are similar to Equation 4-1 with the exception that they are written in matrix form for a small load increment during which the behavior is assumed to be linear elastic.

$$[K]\{\Delta v\} = \{\Delta P\} \qquad \text{(4-136a)}$$

For computational purposes it is convenient to rewrite this equation in the following form

$$[K_t]\{\Delta v\} + \{R_t\} = \{P\} \qquad \text{(4-136b)}$$

where K_t is the tangent stiffness matrix for the current load increment and R_t is the restoring force at the beginning of the load increment which is defined as

$$\{R_t\} = \sum_{i=1}^{n-1} [K_{ti}]\{\Delta v_i\}$$

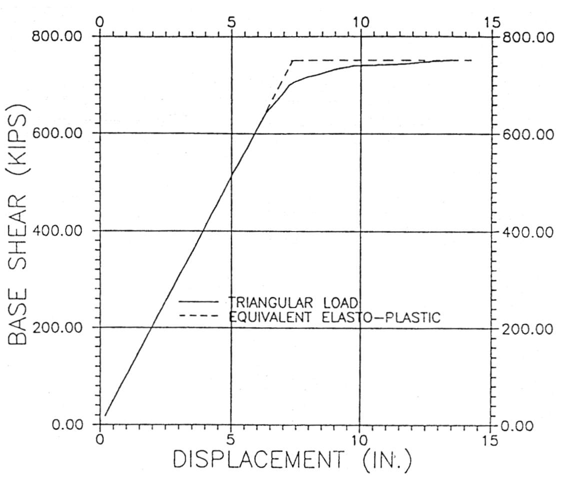

Figure 4-29a. Pushover Curve, Six Story Steel Building.

The lateral force distribution is generally based on the static equivalent lateral forces specified in building codes which tend to approximate the first mode of vibration. These forces are increased in a proportional manner by a specified load factor. The lateral loading is increased until either the structure becomes unstable or a specified limit condition is attained. The results from this type of analysis are usually presented in the form of a graph plotting base shear versus roof displacement. The pushover curve for a six-story steel building [4-18] is shown in Figure 4-29a and the sequence of plastic hinging is shown in Figure 4-29b.

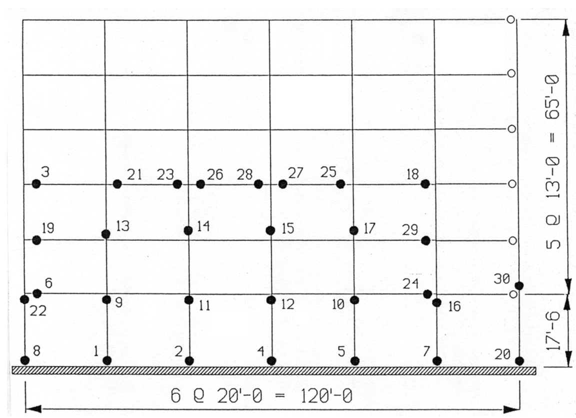

Figure 4-29b Sequence of Plastic Hinge Formation, Six Story Steel Building.

The equations of equilibrium for a multiple degree of freedom system subjected to base excitation can be written in matrix form as

$$\begin{aligned}[M]\{\ddot{v}\} + [C]\{\dot{v}\} + [K]\{v\} \\ = -[M]\{\Gamma\}\ddot{g}(t)\end{aligned} \qquad \text{(Eq.4-137)}$$

This equation is of the same form as that of Eq. 4-76 for the single degree of freedom system. The acceleration, velocity and displacement have been replaced by vectors containing the additional degrees of freedom. The mass has been replaced by the mass matrix which for a lumped mass system is a diagonal matrix with the translational mass and rotational mass terms on the main diagonal. The incremental stiffness has been replaced by the incremental stiffness matrix and the damping has been replaced by the damping matrix. This latter term requires some additional discussion. In the mode superposition method, the damping ratio was defined for each mode of vibration. However, this is not possible for a nonlinear system because it has no true vibration modes. A useful way to define the damping matrix for a nonlinear system is to assume that it can be represented as a linear combination of the mass and stiffness matrices of the initial elastic system

$$[C] = \alpha[M] + \beta[K] \qquad \text{(Eq 4-138)}$$

Where α and β are scaler multipliers which may be selected so as to provide a given percentage of critical damping in any two modes of vibration of the initial elastic system. These two multipliers can be evaluated from the expression

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = 2\begin{bmatrix} \omega_j & -\omega_i \\ -\dfrac{1}{\omega_j} & \dfrac{1}{\omega_i} \end{bmatrix} \frac{\omega_i \omega_j}{\omega_j^2 - \omega_i^2} \begin{Bmatrix} \lambda_i \\ \lambda_j \end{Bmatrix} \qquad \text{(Eq.4-139)}$$

where ω_i and ω_j are the percent of critical damping in the two specified modes. Once the coefficients α and β are determined, the damping in the other elastic modes is obtained from the expression

$$\lambda_k = \frac{\alpha}{2\omega_k} + \frac{\beta \omega_k}{2} \qquad \text{(Eq. 4-140)}$$

A typical damping function which was used for the nonlinear analysis of a reinforced concrete frame [(4-10)] is shown in Figure 4-30. Although the representation for the damping is only approximate it is justified for these types of analyses on the basis that it gives a good approximation of the damping for a range of modes of vibration and these modes can be selected to be the ones that make the major contribution to the response. Also in nonlinear dynamic analyses the dissipation of energy through inelastic deformation tends of overshadow the dissipation of energy through viscous damping. Therefore, an exact expresentation of damping is not as important in a nonlinear system as it is in a linear system. One should be aware of the characteristics of the damping function to insure that important components of the response are not lost. For instance, if the coefficients are selected to give a desired percentage of critical damping in the lower modes and the response of the higher modes is important, the higher mode response may be over damped and its contribution to the total response diminished.

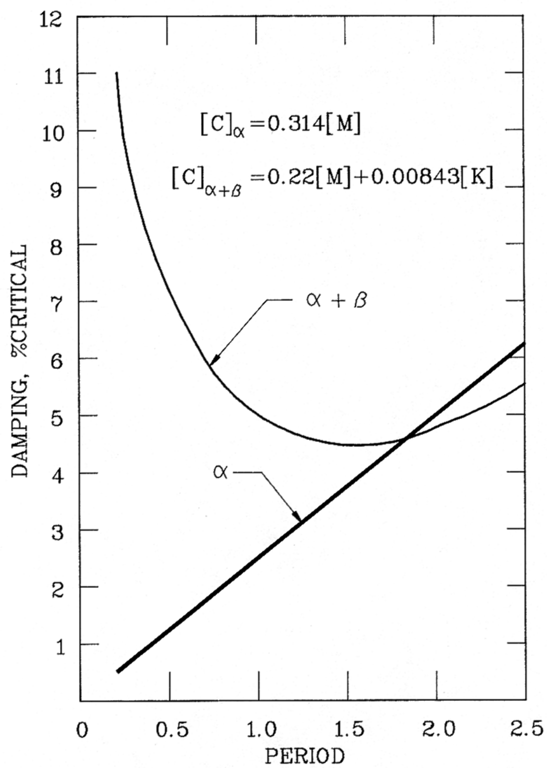

Figure 4-30. Damping functions for a framed tube.

Substituting Eq. 4-138 into Eq. 4-137 results in

$$[M]\{\ddot{v}\} + \alpha[M]\{\dot{v}\} + \beta[K_i]\{\dot{v}\} + [K]\{v\} = -[M]\{\Gamma\}\ddot{g}(t) \qquad \text{(Eq. 4-141)}$$

where K_i refers to the initial stiffness.

Representing the incremental stiffness in terms of the tangent stiffness, K_t, and rearranging some terms, results in

$$[K]\{v\} = [K_t]\{\Delta v\} = \{R_t\} + [K_t]\{\Delta v\} \quad \text{(Eq. 4-142)}$$

where

$$\{R_t\} = \sum_{i=1}^{n-1} [K_{ti}]\{\Delta v_i\}$$

Using the step-by-step integration procedure in which the acceleration is assumed to be constant during a time increment, equations similar to Eqs. 4.84 and 4-86 can be developed for the multiple degree of freedom system which express the acceleration and velocity vectors at the end of the time increment in terms of the incremental displacement vector and the vectors of initial conditions at the beginning of the time increment:

$$\{\ddot{v}(t)\} = (\frac{4}{\Delta t^2})\{\Delta v\} + \{A_t\} \quad \text{(Eq. 4-143)}$$

$$\{\dot{v}(t)\} = (\frac{2}{\Delta t})\{\Delta v\} + \{B_t\} \quad \text{(Eq. 4-144)}$$

$$\{v(t)\} = \{v(t-\Delta t)\} + \{\Delta v\} \quad \text{(Eq. 4-144a)}$$

where

$$\{A_t\} = -\frac{4}{\Delta t}\{\dot{v}(t-\Delta t)\} - \{\ddot{v}(t-\Delta t)\} \quad \text{(Eq. 4-145)}$$

$$\{B_t\} = -\{\dot{v}(t-\Delta t)\} \quad \text{(Eq. 4-146)}$$

Substituting Eqs. 4-142 through 4-146 into Eq. 4-141 and rearranging some terms leads to the pseudo-static form

$$[\tilde{K}]\{\Delta v\} = \{\tilde{P}\} \quad \text{(Eq. 4-147)}$$

where

$$[\tilde{K}] = \left[C_0[M] + C_1[K_i] + [K_t]\right]$$

$$\{\tilde{P}\} = \{P(t)\} - \{R_t\} - [M]\left\{\{A_t\} + \alpha\{B_t\}\right\} - \beta[K_i]\{B_t\}$$

$$C_0 = \frac{4}{\Delta t^2} + \frac{2\alpha}{\Delta t}$$

$$C_1 = \frac{2\beta}{\Delta t}$$

The incremental displacement vector can be obtained by solving Eq. 4-147 for $\{\Delta v\}$ This result can then be used in Eqs. 4-143, 4-144 and 4-144a to obtain the acceleration vector, the velocity vector and the displacement vector at the end of the time interval. These vectors then become the initial conditions for the next time interval and the process is repeated.

Output from a nonlinear response analysis of a MDOF system generally includes response parameters such as the following: an envelope of the maximum story displacements, an envelope of the maximum relative story displacement divided by the story height (sometimes referred to as the interstory drift index (IDI), an envelope of maximum ductility demand on structural members such as beams, columns, walls and bracing, an envelope of maximum rotation demand at the ends of members, an envelope of the maximum story shear, time history of base shear, moment versus rotation hysteresis plots for critical plastic hinges, time history plots of story displacements and time history plots of energy demands (input energy, hysteretic energy, kinetic energy and dissapative energy).

For multiple degree of freedom systems, the definition of ductility is not as straight-forward as it was for the single degree of freedom systems. Ductility may be expressed in terms of such parameters as displacement, relative displacement, rotation, curvature or strain.

Example 4-10.Seismic Response Analyses

The following is a representative response analysis for a six story building in which the lateral resistance is provided by moment resistant steel frames on the perimeter. The structure has a rectangular plan with typical dimensions of $228' \times 84'$ as shown in Figure 4-31. The building was designed for the requirements of the 1979 Edition of the Uniform Building Code (UBC) with the seismic load based on the use of static equivalent lateral forces.

Elastic Analyses

As a first step in performing the analyses, the members of the perimeter frame will be stress checked for the design loading conditions and the dynamic properties of the building will be determined. This will help to insure that the analytical model of the building is correct and that the gravity loading which will be used for the nonlinear response analysis is also reasonable. This will be done using a three dimensional model of the lateral force system and the ETABS [(4-11)] computer program. This program is widely used on the west coast for seismic analysis and design of building systems. An isometric view of the perimeter frame including the gravity load is shown in Figure 4-32. The location of the concentrated and distributed loads depends upon the framing system shown in Figure 4-31.

Using the post-processor program STEELER [(4-12)], the lateral force system is stress checked using the AISC-ASD criteria. The stress ratio is calculated as the ratio of the actual stress in the member to the allowable stress. Applying the gravity loads in combination with the static equivalent lateral forces in the transverse direction produces the stress ratios shown in Figure 4-33. This result includes the effect of an accidental eccentricity which is 5% of the plan dimension. The maximum stress ratio in the columns is 0.71 and the maximum in the beams is 0.92. These values are reasonable based on standard practice at the time the building was designed. Ideally, the stress ratio should be just less than one, however, this is not always possible due to the finite number of steel sections that are available.

Modal analyses indicate that the first three lateral modes of vibration in each direction represent more than 90% of the participating mass. In the transverse direction, these modes have periods of vibration of 1.6, 0.6 and 0.35 seconds. In the longitudinal direction, the periods are slightly shorter.

Dynamic analyses are conducted using the same analytical model and considering an ensemble of five earthquake ground motions recorded during the Northridge earthquake. A representative time history of one of these motions is shown in Figure 4-34. The corresponding stress ratios in the perimeter frame are shown in Figure 4-35 for earthquake motion applied in the transverse direction. Stress ratios in the beams of the transverse frames range from 2.67 to 4.11 indicating substantial inelastic behavior. Stress ratios in excess of 1.12 are obtained in all of the columns of the transverse frames, however, it should be recalled that there is a factor of safety of approximately 1.4 on allowable stress and plastic hinging.

Nonlinear Analyses

In order to estimate the lateral resistance of the building at ultimate load, a static, nonlinear analysis (pushover) is conducted for proportional loading. The reference lateral load distribution is that specified in the 1979 UBC. This load distribution is then multiplied by a load factor to obtain the ultimate load. The nonlinear model is a two dimensional model in which the plasticity is assumed to be concentrated in plastic hinges at the ends of the members.

The results of the pushover analysis are usually represented in terms of a plot of the roof displacement versus the base shear as shown in Figure 4-36. This figure indicates that first yielding occurs at a base shear of approximately 670 kips and a roof displacement of approximately 7.25 inches. The UBC 1979 static equivalent lateral forces for this frame results in a base shear of 439 kips which implies a load factor of 1.52 on first yield. At a roof displacement of 17.5 inches, a sway mechanism forms with all girders hinged and

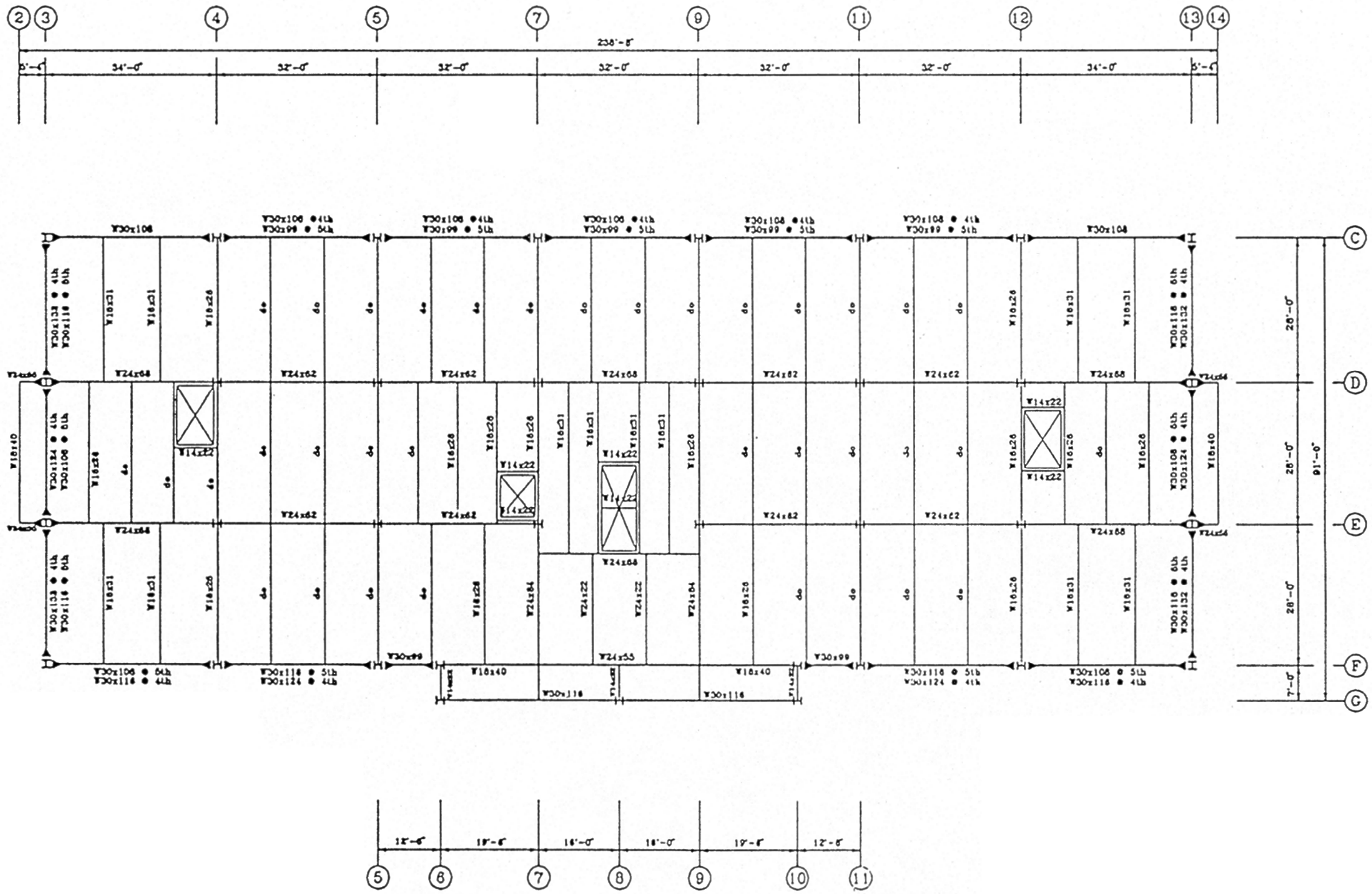

Figure 4-31 Typical floor framing plan ~ Fourth & fifth floors

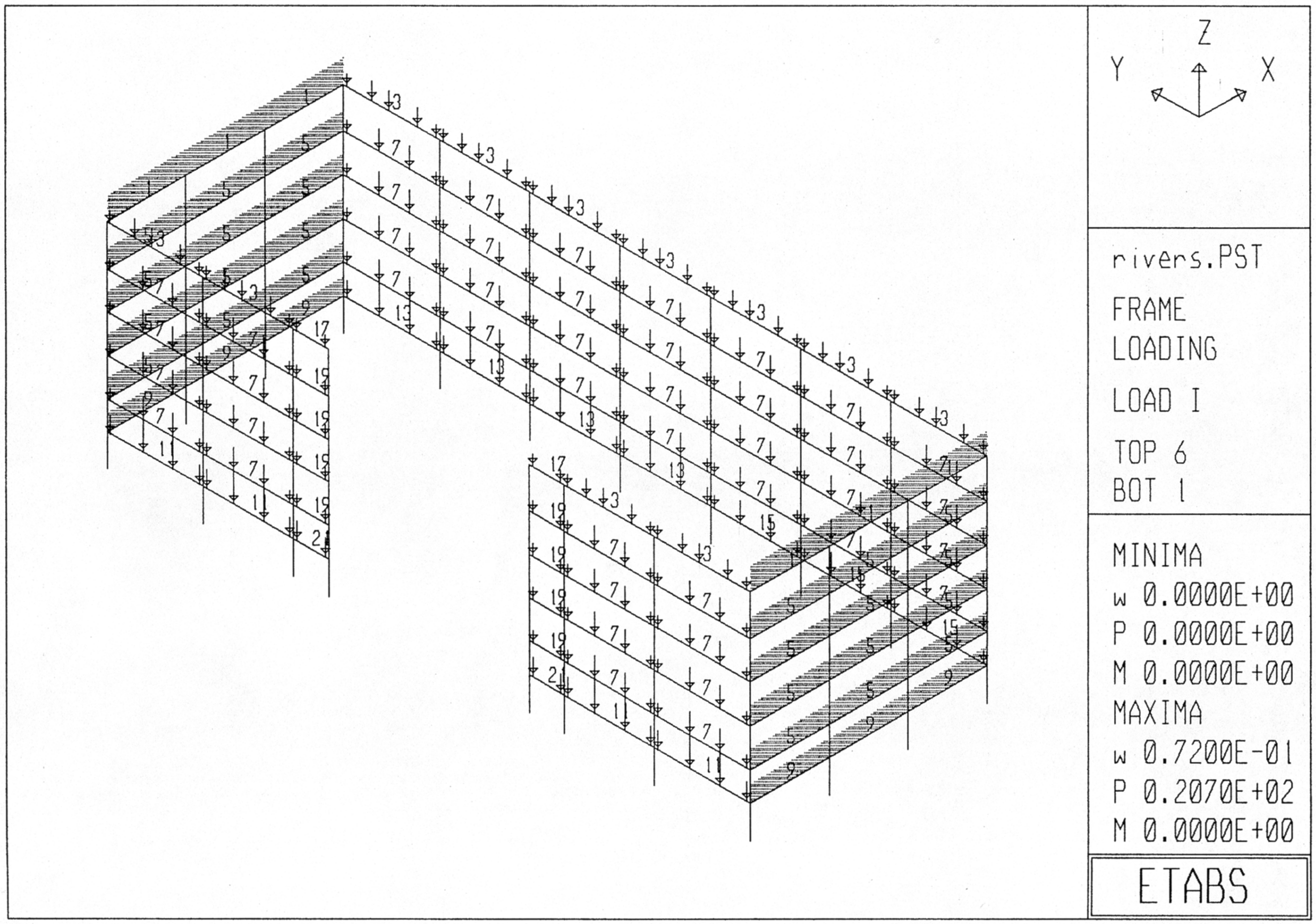

Figure 4-32. Gravity Loading Pattern, ETABS

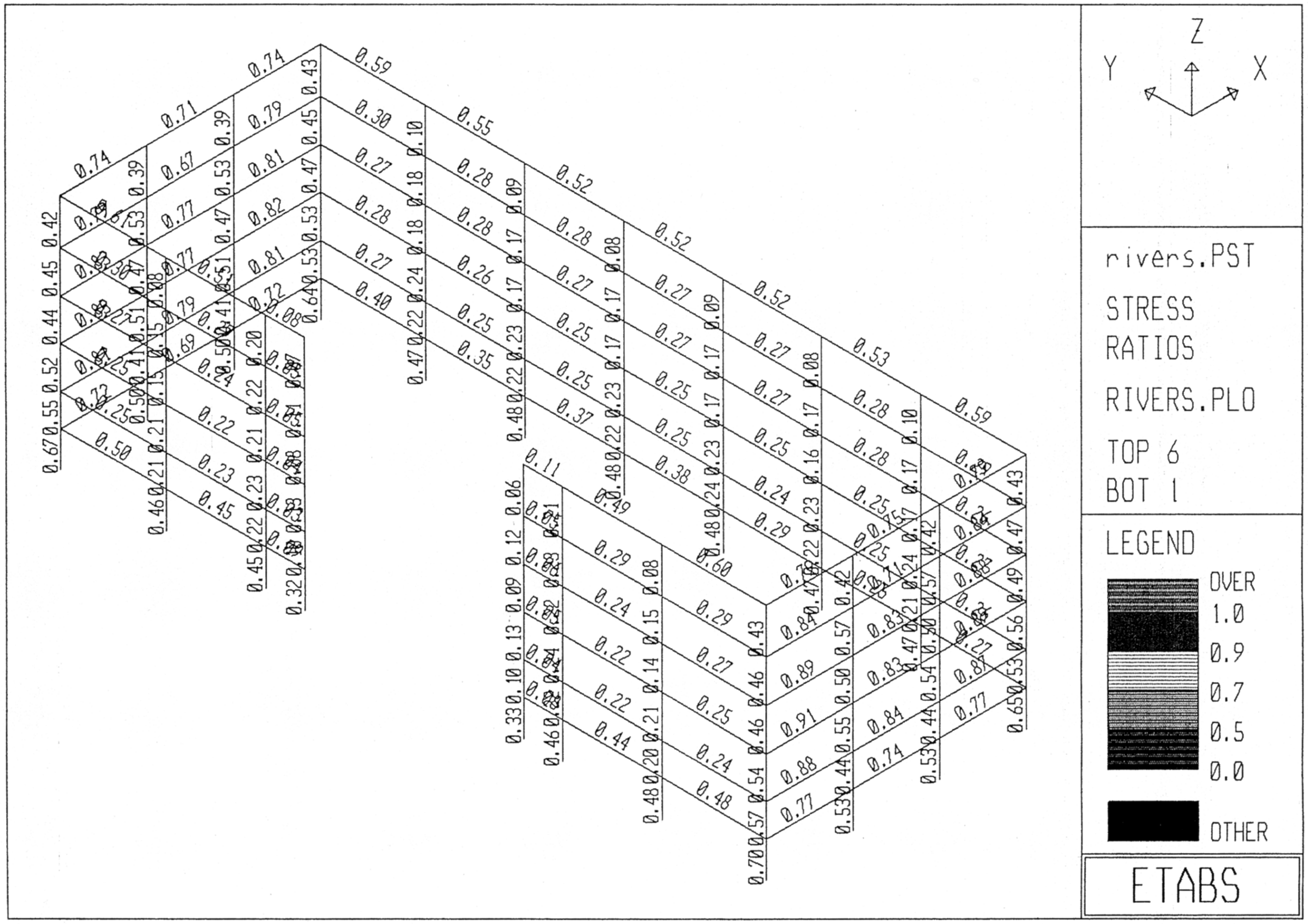

Figure 4-33. Calculated Stress Ratios, Design Loads, ETABS

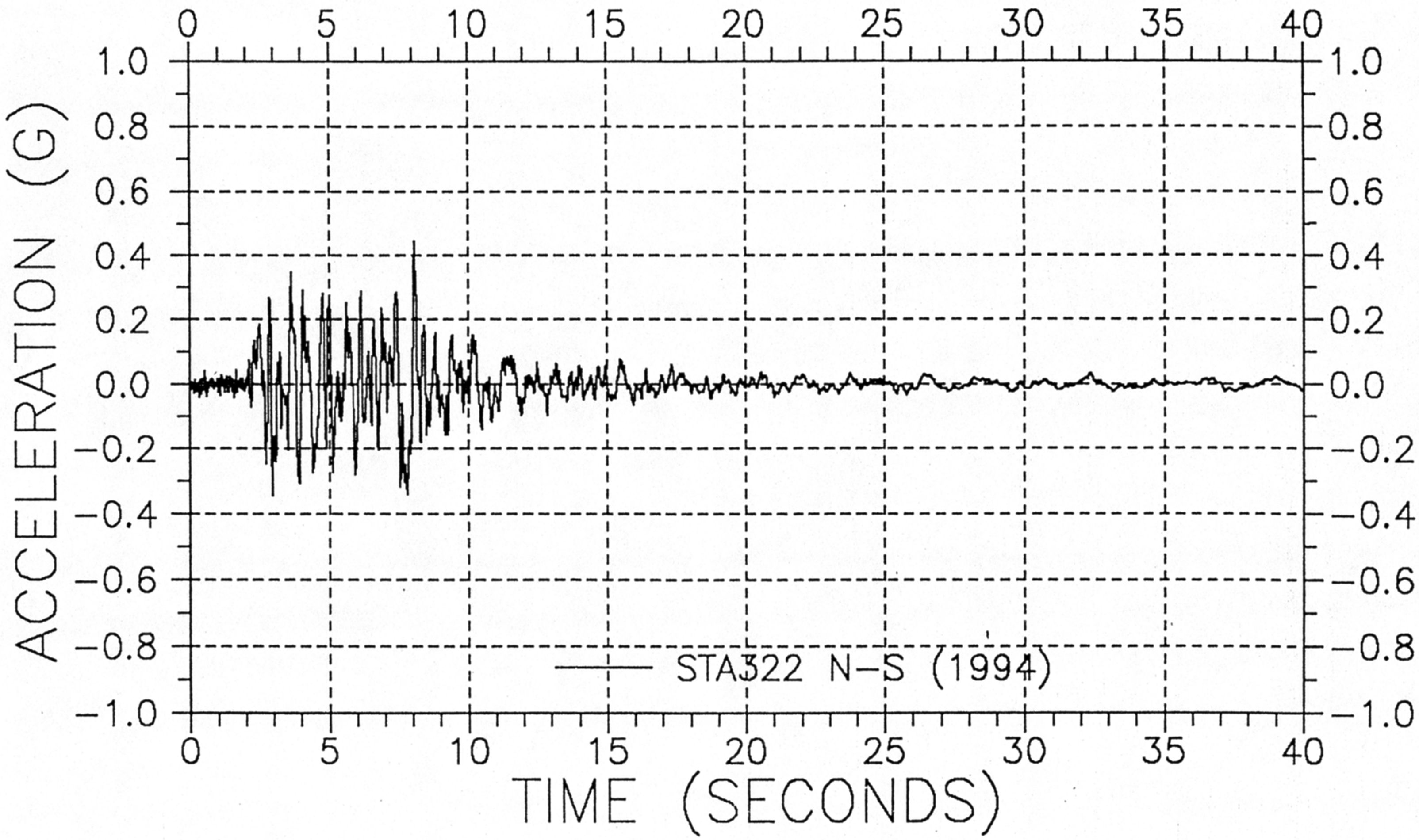

Figure 4-34. Recorded Base Acceleration, Sta. 322, N-S

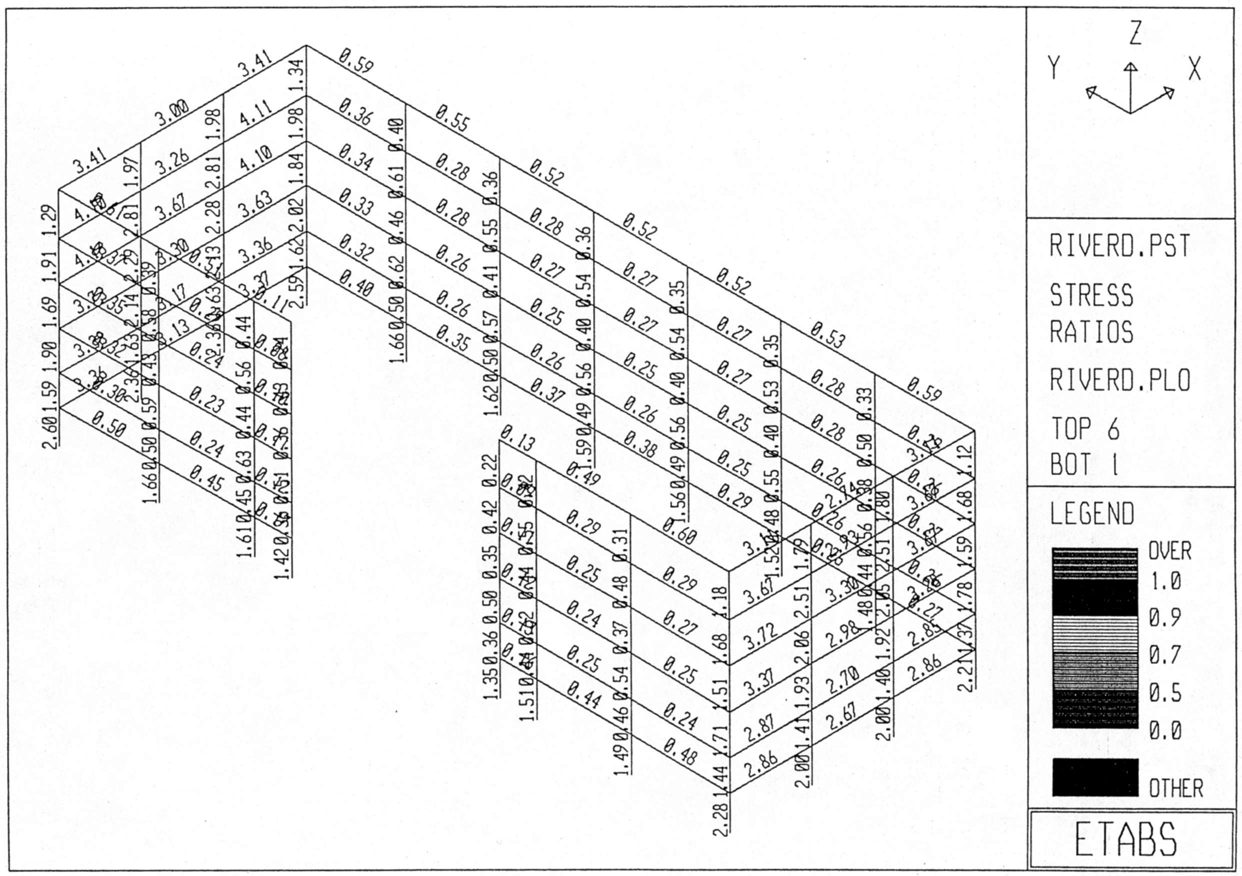

Figure 4-35. Calculated Stress Ratios, Sta. 322 Ground Motion

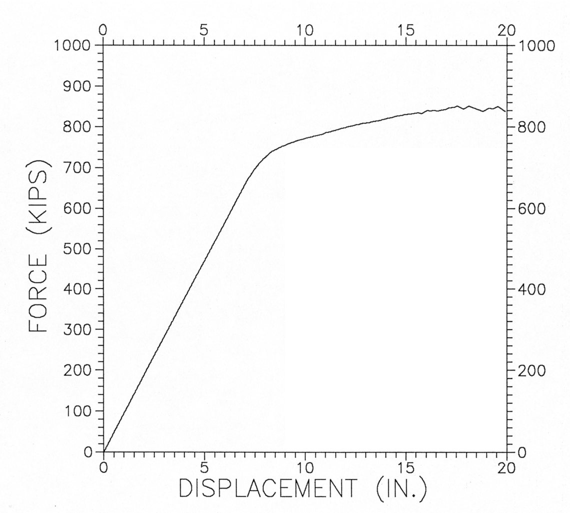

Figure 4-36. Static Pushover Curve

hinges at the base of the columns. At this displacement, the pushover curve is becoming almost horizontal indicating a loss of most of the lateral stiffness. This behavior is characterized by a large increase in displacement for a small increase in lateral load since lateral resistance is only due to strain hardening in the plastic hinges. The ultimate load is taken as 840 kips which divided by the code base shear for the frame (439 kips) results in a load factor of 1.91 on ultimate.

Note that the elastic dynamic analysis for the acceleration shown in Figure 4-34 results in a displacement at the roof of 16.7 inches. Comparing this to the pushover curve (Figure 4-36) indicates that the structure should be well into the inelastic range based on the displacement response.

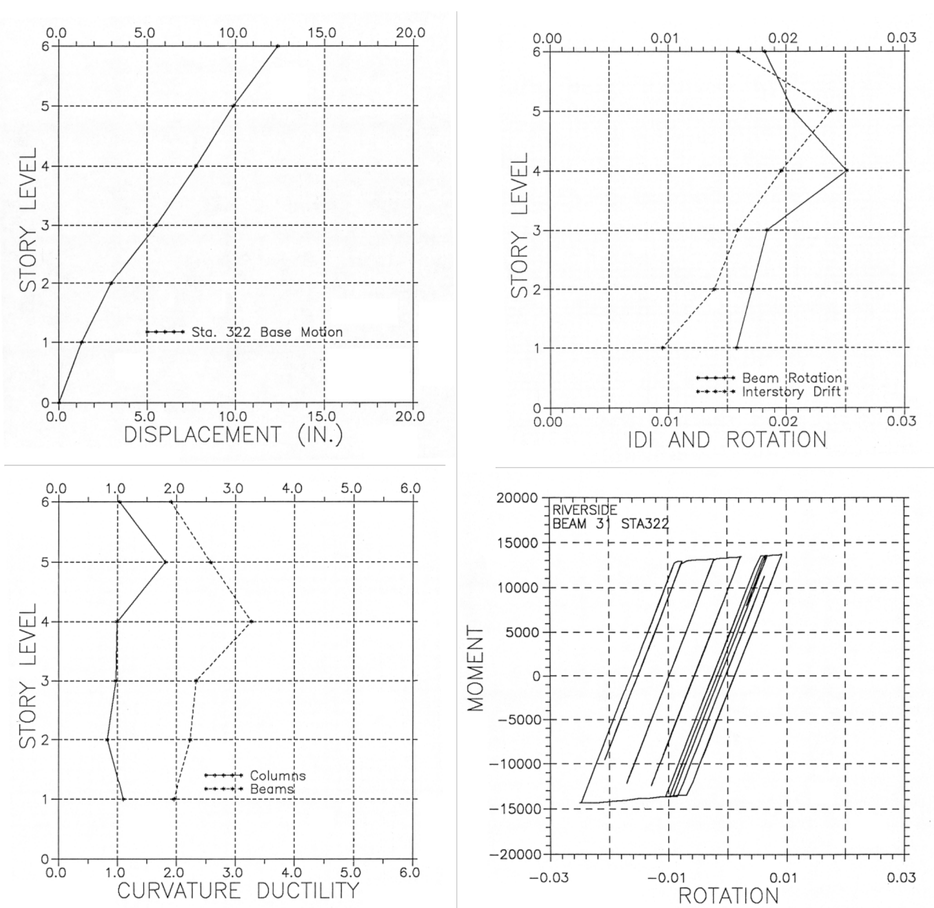

Figure 4-37. Calculated Nonlinear Dynamic Response.

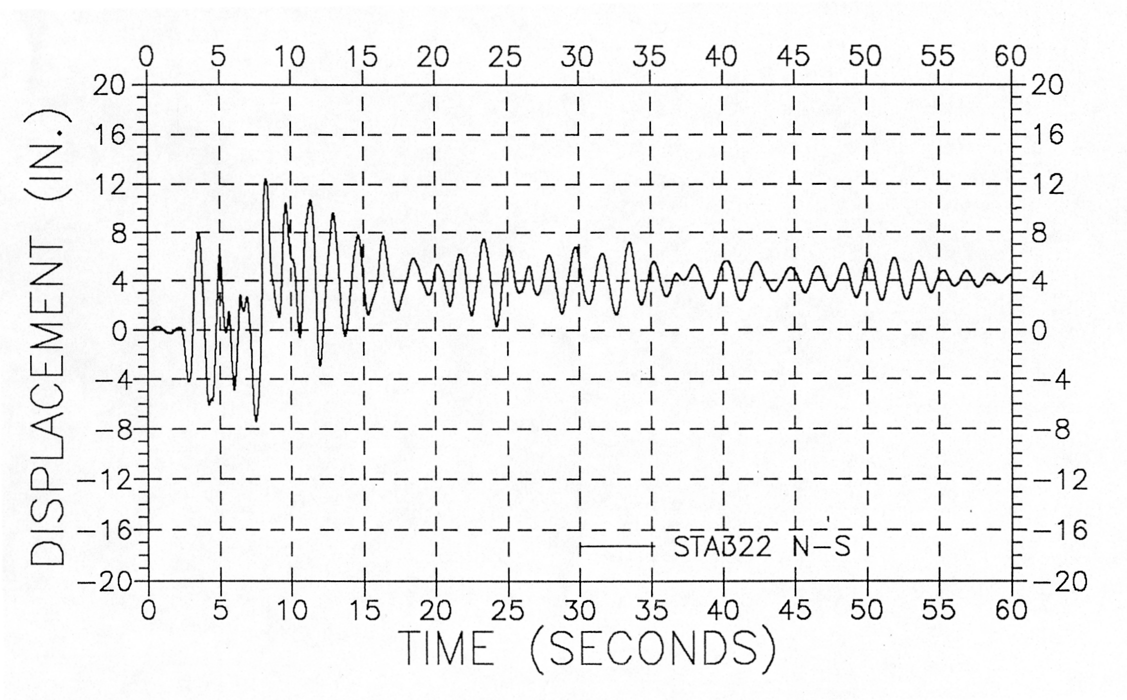

Figure4-38. Nonlinear Displacement, Roof Level

The nonlinear dynamic response of a structure is often presented in terms of the following response parameters: (1) envelope of maximum total displacement, (2) envelope of maximum story to story displacement divided by the story height (interstory drift index), (3) maximum ductility demand for the beams and columns, (4) envelopes of maximum plastic hinge rotation, (5) moment versus rotation hysteresis curves for critical members and (6) envelopes of maximum story shear. Representative plots of four of these parameters are shown in Figure 4-37. The lateral displacement envelope (Figure 4-37a) indicates that the maximum displacement at the roof level is 12.3 inches which is less than the 16.7 inches obtained from the elastic dynamic analysis. The interstory drift and total beam rotation curves are shown in Figure 4- 37b which indicates that the interstory drift ranges from 0.01 (1%) to 0.024 (2.4%). The beam rotation can be seen to range between 0.016 and 0.025. The curvature ductility demands of the beams and columns is shown in Figure 4-37c. The maximum ductility demand for the columns is 1.8 and for the beams it is 3.3. The hysteretic behavior of a plastic hinge in a critical beam is shown in a plot of moment versus rotation in Figure 4- 37d.

A final plot, Figure 4-38, shows the nonlinear displacement time history of the roof. This figure illustrates the displacement of a pulse type of input. After some lessor cycles during the first 7 seconds of the time history, the structure sustains a strong displacement at approximately 8 seconds which drives the roof to a displacement of 12 inches relative to the base. Note the acceleration pulse at this time in the acceleration time history (Figure 4-34). Following this action, the structure begins to oscillate about a new, deformed position at four inches displacement. This is a residual displacement, which the structure will have following the earthquake and is characteristic of inelastic behavior. Additional details of this analysis example can be found in the literature [(4-13)].

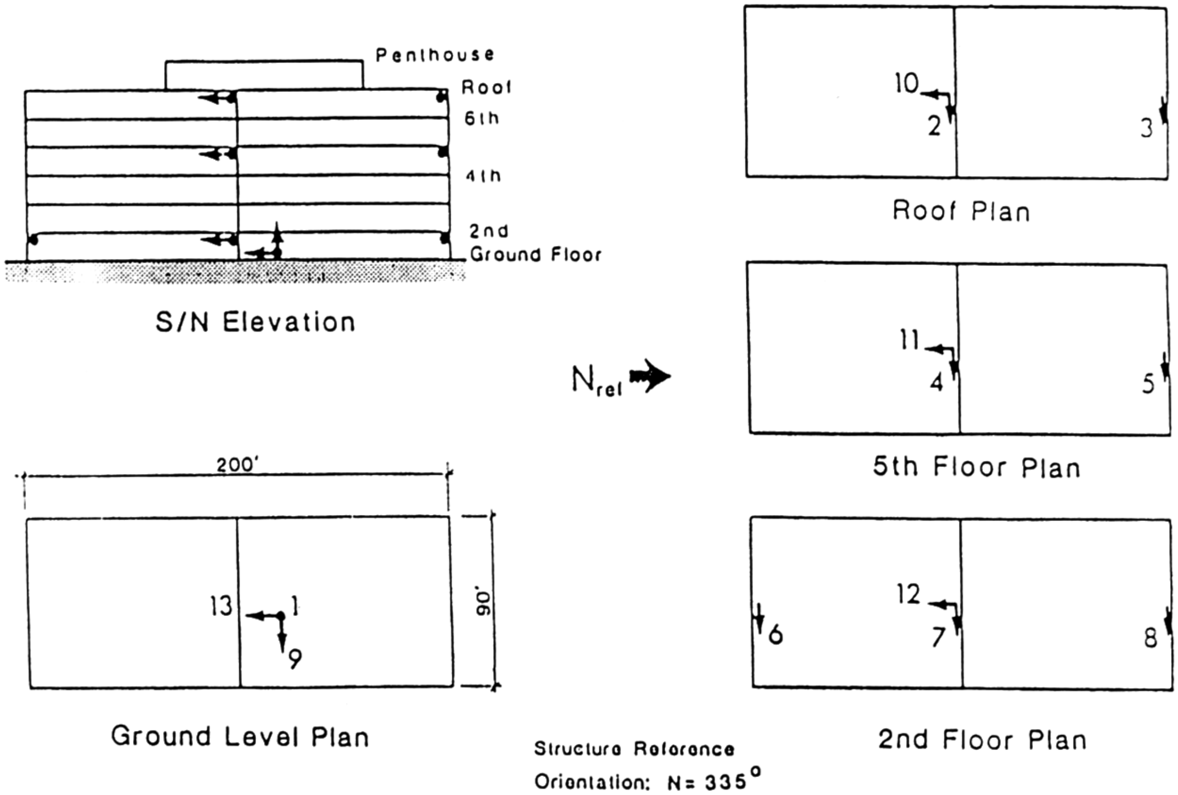

Figure 4-39. Location of Strong Motion Instrumentation

4.8 VERIFICATION OF CALCULATED RESPONSE

The dynamic response procedures discussed in the previous sections must have the ability to reliably predict the dynamic behavior of structures when they are subjected to critical seismic excitations. Hence, it is necessary to compare the results of analytical calculations with the results of large-scale experiments. The best large-scale experiment is when an earthquake occurs and properly placed instruments record the response of the building to ground motions recorded at the base. The instrumentation (accelerometers) placed in a six-story reinforced concrete building by the California Strong Motion Instrumentation Program (CSMIP) is indicated in Figure 4-39. The lateral force framing system for the building, shown in Figure 4-40, indicates that there are three moment frames in the transverse (E-W) direction and two moment frames in the longitudinal (N-S) direction. Note that the transverse frames at the ends of the building are not continuous with the longitudinal frames. It is assumed that the floor diaphragms are rigid in their own plane. During the Loma Prieta earthquake the instrumentation recorded thirteen excellent records of building response having a duration of more than sixty seconds [4-19]. Since the response was only weakly nonlinear, the calculations can be made using the ETABS program, however, similar analyses can also be conducted with a nonlinear response program [4-20].

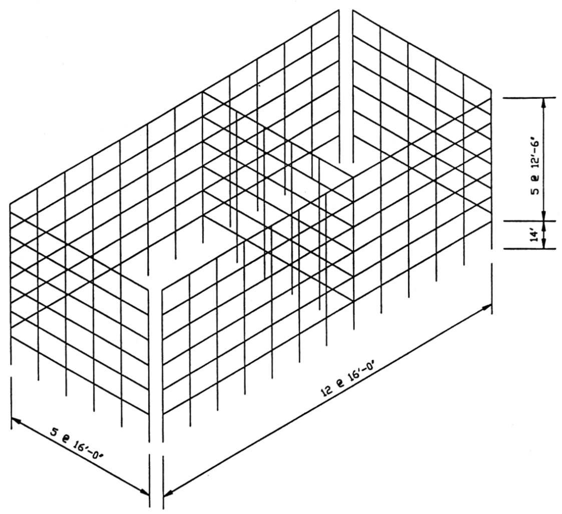

Figure 4-40. ETABS Building Model

To improve the evaluation of the recorded response, spectral analyses are conducted in both the time domain (response spectra) and frequency domain (Fourier spectra). A further refinement of the Fourier analysis can be attained by calculating a Fourier amplitude spectra for a segment (window) of the recorded time history. The fixed duration window is then shifted along the time axis and the process is repeated until the end of the time history record. This results in a "moving window Fourier amplitude spectra" (MWFAS) which indicates the changes in period of the building response during the time history as shown in Figure 4-41. In this example a ten-second window was used with a five-second shift for the first sixty seconds of the recorded response. In general, the length of the "window" should be at least 2.5 times the fundamental period of the structure.

If the connections (offsets) are assumed to be rigid, the initial stiffness of the building prior to any cracking of the concrete can be estimated using the analytical model with member properties of the gross sections. This results in a period of 0.71 seconds in the E-W direction and 0.58 seconds in the N-S direction. This condition can also be evaluated by the results obtained from the initial window of the MWFAS. An examination of Figure 4-41 indicates an initial period of 0.71 in the E-W direction and 0.58 seconds in the N-S direction. Identical results were also obtained from ambient vibration tests conducted by Marshall, et al. [4-21].

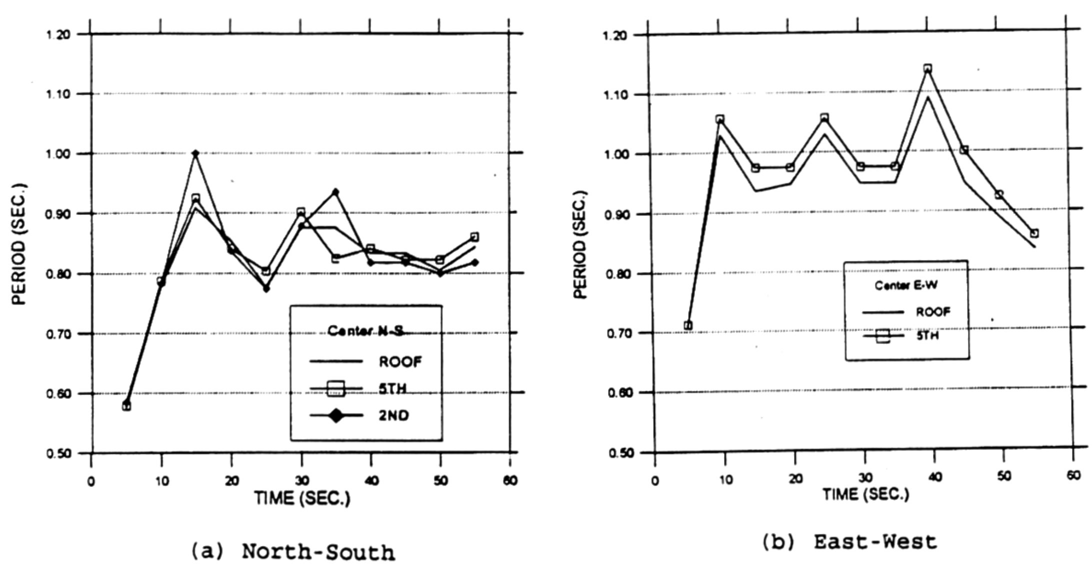

Figure 41. Variation of Building Period with Time

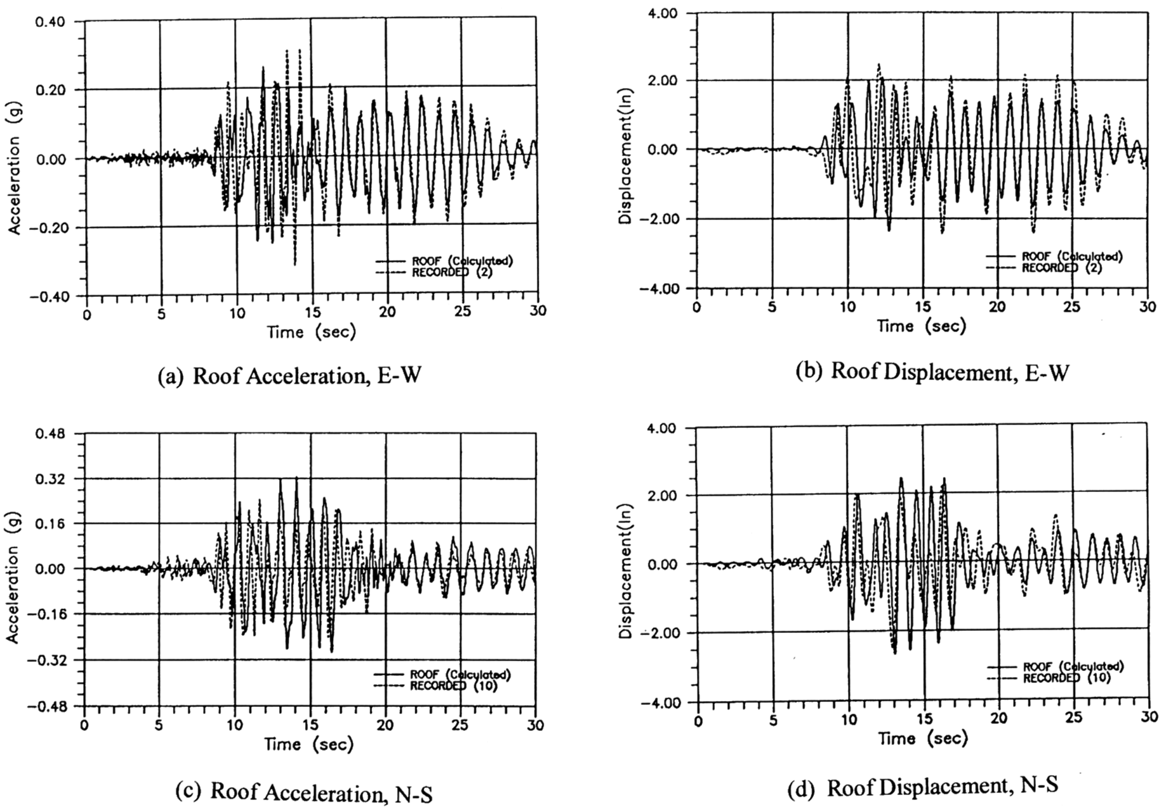

Figure 4-42. Time History Comparisons of Acceleration, Displacement

During the strong motion portion of the response, cracking in the concrete and limited yielding of the tension steel will cause the period of vibration to lengthen. In order to represent this increased flexibility in the elastic analytical model, the flexibility of the individual members can be reduced to an effective value or the rigid offsets at the connections [(4-13)] can be reduced in length. For this example, the rigid offsets were reduced by fifty percent. This results in a period of 1.03 seconds in the E-W direction and 0.89 seconds in the N-S direction which are in the range of values obtained from the MWFAS. Considering the entire duration of the recorded response, the Fourier amplitude spectra indicates a period of 1.05 seconds in the E-W direction and 0.85 in the N-S direction. Corresponding values obtained from a response spectrum analysis indicate 1.0 E-W and 0.90 N-S. It can be concluded that for this building, all of these values are in good agreement. The MWFAS also indicate an increase in period of approximately fifty percent in both principal directions during the earthquake. This amount of change is not unusual for a reinforced concrete building [(4-22)], however, it does indicate cracking and possible limited yielding of the reinforcement. The time histories of the acceleration and displacement at the roof level are shown in Figure 4-42. This also shows a good correlation between the measured and the calculated response.

REFERENCES

4-1 Chopra, A, K., *Dynamics of Structures, A Primer*, Earthquake Engineering Research institute, Berkeley, CA, 1981.

4-2 Applied Technology Council, *An Evaluation of a Response Spectrum Approach to Seismic Design of Buildings*, ATC-2, Applied Technology Council, Palo Alto, CA, 1974.

4-3 Newmark, N. M., and Hall, W. J., "Procedures and Criteria for Earthquake Resistant Design", Building Practices for Disaster Mitigation, U. S. Department of Commerce, Building Science Series 46, 1973.

4-4 Clough, R. W., and Penzien, J., *Structural Dynamics*, McGraw-Hill, Inc., New York, NY, 1975.

4-5 Rayleigh, Lord, *Theory of Sound*, Dover Publications, New York, NY, 1945.

4-6 Lee, V. W., and Trifunac, M. D., "Strong Earthquake Ground Motion Data in EQINFOS, Part I," Report No.87-01, Department of Civil Engineering, USC, Los Angeles, CA, 1987.

4-7 Bathe, K. J., and Wilson, E. L., *Numerical Methods in Finite Element Analysis*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1976.

4-8 Caughy,T. K., "Classical Normal Modes in Damped Linear Dynamic systems," Journal of Applied Mechanics, ASME, Paper No. 59-A-62, June, 1960.

4-9 Wilson, E. L., Der Kiureghian, A.Rand Bayo, E. P., "A Replacement for the SRSS Method in Seismic Analysis", Earthquake Engineering and Structural Dynamics, Vol. 9, 1981.

4-10 Anderson, J. C., and Gurfinkel, G., "Seismic Behavior of Framed Tubes," International Journal of Earthquake Engineering and Structural Dynamics, Vol. 4, No. 2, October-December, 1975.

4-11 Habibullah, A., "ETABS, Three Dimensional Analysis of Building Systems," User's Manual, Version 6.0, Computers & Structures, Inc. Berkeley, California, 1994.

4-12 Habibullah, A., "ETABS Design Postprocessors," Version 6.0, Computers & Structures, Inc. Berkeley, California, 1994.

4-13 Anderson, J. C., "Moment Frame Building", Buildings Case Study Project, SSC 94-06, Seismic Safety Commission, State of California, Sacramento, California, 1996.

4-14 Charney, F.A., NONLIN, Nonlinear Dynamic Time History Analysis of Single Degree of Freedom Systems, Advanced Structural Concepts, Golden, Colorado, 1996.

4-15 Norris, C.H., Hansen, R.J., Holley, M.J., Biggs, J.M., Namyet, S., and Minami, J.K., Structural Design for Dynamic Loads, McGraw-Hill Book Company, New York, New York, 1959.

4-16 U.S. Army Corps of Engineers, *Design of Structures to Resist the Effects of Atomic Weapons*, EM 1110-345-415, 1957.

4-17 Applied Technology Council, "Seismic Evaluation and Retrofit of Concrete Buildings", ATC-40, Applied Technology Council, Redwood City, California, 1991.

4-18 Anderson, J.C. and Bertero, V.V., "Seismic Performance of an Instrumented Six Story Steel Building", Report No. UCB/EERC-91/111, Earthquake Engineering Research Center, University of California at Berkeley, Berkeley, California, 1991.

4-19 California Division of Mines and Geology, "CSMIP Strong-Motion Records from the Santa Cruz Mountains (Loma Prieta) California Earthquake of October 17, 1989", Report OSMS 89-06.

4-20 Anderson, J.C. and Bertero, V.V., "Seismic Performance of an Instrumented Six Story Steel Building", Report No. UCB/EERC-93/01, Earthquake Engineering Research Center, University of California at Berkeley.

4-21 Marshall, R.D., Phan, L.T. and Celebi, M., "Full Scale Measurement of Building Response to Ambient Vibration and the Loma Prieta Earthquake", Proceedings, Fifth National Conference on Earthquake Engineering, Vol. II, Earthquake Engineering Research Institute, Oakland, California, 1994.

4-22 Anderson, J.C., Miranda, E. and Bertero, V.V., "Evaluation of the Seismic Performance of a Thirty-Story RC Building", Report No. UCB-EERC-93/01, Earthquake Engineering Research Center, University of California, Berkeley.